NON-CONVEX SHREDDED SIGNAL RECONSTRUCTION VIA SPARSITY ENHANCEMENT

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1. Motivation

- Recovering lost information by reconstruction of shredded documents is one of the interesting fields of research in forensic and investigative sciences.
- Shredding of documents is often practiced to destroy potentially incriminating evidences.
- Restoration of shredded signals remains a relevant and significant challenge in archaeological and forensic efforts.
- We present a generic, efficient non-convex optimization method that employs iterative sparsity enhancement of the observed signal.
- A key assumption: most natural signals are sparse in a given representation domain.

2. Shredded Documents and the Problem with Reconstruction

- The documents are typically shredded by using mechanical shredding devices producing thin strips often termed as 'spaghetti' or smaller rectangular pieces, or circular fragments named as 'confetti' or hexagons.
- The problem of shredded document recovery requires enormous amount of time and effort if done manually.

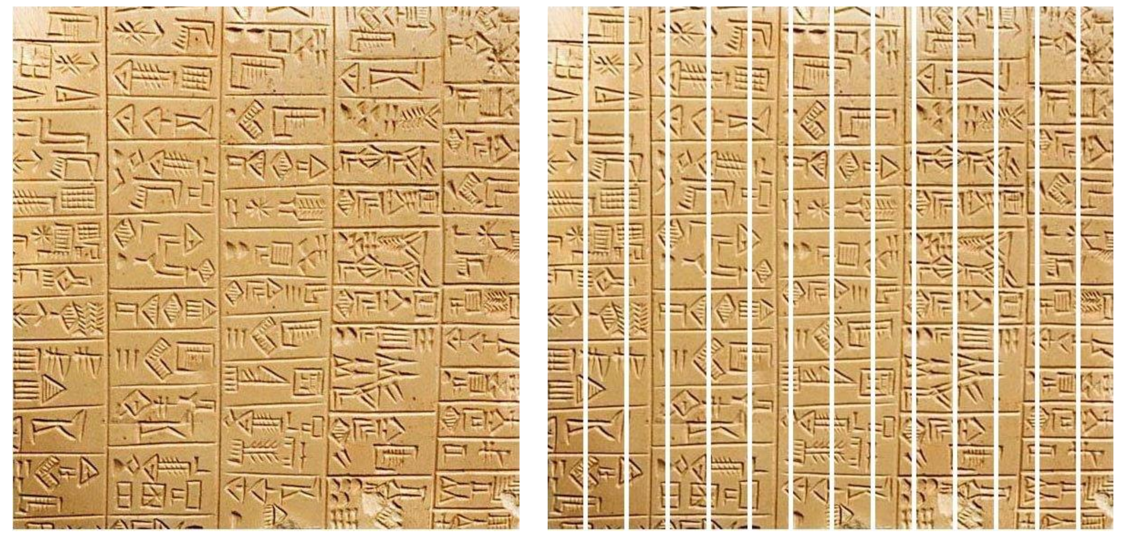


Fig. 1. Rectangular strip-shredded Sumerian inscription: (a) original and (b) 'spaghetti' fragments.

3. Problem Formulation

Goal: Reconstruction of a finite-length discrete-time signal $\boldsymbol{x} \in \mathbb{C}^{MN}$. where M and N are the number of shredded parts and the length of each part, respectively.

The shredded signal be,

$$\boldsymbol{y} = (\boldsymbol{y}_1^T \;\; \boldsymbol{y}_2^T \;\; \boldsymbol{y}_3^T \dots \;\; \boldsymbol{y}_m^T)^T$$

Assumption: The original signal $\boldsymbol{x}$ is sparse in a given representation domain such as the Discrete Fourier Transform (DFT) domain. Let $\boldsymbol{v} \in \mathbb{C}^{MN}$ be the representation of $\boldsymbol{x}$ in the DFT domain as,

$$\boldsymbol{x} = \boldsymbol{\Psi}^H \boldsymbol{v} \qquad (1)$$

where $\boldsymbol{v}$ belongs to the set of all vectors with at most s non-zero values: χ_s for $s \ll MN$ and $\boldsymbol{\Psi}$ is the DFT matrix given by,

$$[\boldsymbol{\Psi}]_{l,p} = \frac{1}{\sqrt{MN}} \exp\left\{\frac{j2\pi lp}{MN}\right\}, \quad l, p = 1,2,\dots,MN.$$

The desired signal partitions $\{\boldsymbol{x}_m\}$ can be obtained via a permutation of $\{\boldsymbol{y}_m\}$ through the permutation matrix $\boldsymbol{P} \in \mathbb{R}^{M\times M}$, viz.

$$\boldsymbol{x} = (\boldsymbol{P} \otimes \boldsymbol{I}_N)\boldsymbol{y} \qquad (2)$$

From (1) and (2), the final optimization problem becomes,

$$\begin{aligned} \min_{\boldsymbol{P},\boldsymbol{v}} & \|(\boldsymbol{P}\otimes\boldsymbol{I}_N)\boldsymbol{y} - \boldsymbol{\Psi}^H\boldsymbol{v}\|_2 \\ \text{s.t} \quad & \boldsymbol{P} \text{ is a permutation matrix of size } M, \\ & \boldsymbol{v} \in \chi_s, \\ & \|\boldsymbol{v}\|_2 = \|\boldsymbol{y}\|_2, \end{aligned} \qquad (3)$$

while s remains unknown.

4. Reconstruction Approach

Observation 1: For given s, (3) can be tackled using cyclic minimization.

- For fixed $\boldsymbol{P}$:

$$\begin{aligned} \min_{\boldsymbol{v}} & \|\boldsymbol{v} - \boldsymbol{\Psi}(\boldsymbol{P}\otimes\boldsymbol{I}_N)\boldsymbol{y}\|_2 \\ \text{s.t.} \quad & \boldsymbol{v} \in \chi_s, \\ & \|\boldsymbol{v}\|_2 = \|\boldsymbol{y}\|_2. \end{aligned} \qquad (4)$$

Let $\tilde{\boldsymbol{v}} = \boldsymbol{\Psi}(\boldsymbol{P}\otimes\boldsymbol{I}_N)\boldsymbol{y}$ such that, $\|\boldsymbol{v} - \tilde{\boldsymbol{v}}\|_2^2 = c - 2\boldsymbol{v}^T\tilde{\boldsymbol{v}}$ where $c = \|\boldsymbol{v}\|_2^2 + \|\tilde{\boldsymbol{v}}\|_2^2$ is constant.

The optimal $\boldsymbol{v}$ can be given as

$$\boldsymbol{v}_{opt} = \|\boldsymbol{y}\|_2 \left(\frac{\tilde{\boldsymbol{v}}\odot\boldsymbol{\mu}}{\|\tilde{\boldsymbol{v}}\odot\boldsymbol{\mu}\|_2}\right) \qquad (5)$$

- For fixed $\boldsymbol{v}$: (3) can be written as

$$\min_{\boldsymbol{P}} \sum_{m=1}^{M}\sum_{l=1}^{N}\left|\sum_{k=1}^{M} p_{m,k}\cdot y_{k,l} - \hat{v}_{m,l}\right|^2 \qquad (6)$$

where $\hat{\boldsymbol{v}} = \boldsymbol{\Psi}^H\boldsymbol{v}$. As $\boldsymbol{P}$ only consists of {0,1} values, we have, $\sum_{k=1}^{M} p_{m,k}\cdot y_{k,l} = y_{\pi_{\bar{m}}}$ where $\pi_{\bar{m}}$ is the only column in $\bar{m}$th row of matrix $(\boldsymbol{P}\otimes\boldsymbol{I}_N)$ where the respective entry is 1. Hence, the optimization problem can simply be written as,

$$\min_{\{\pi_{\bar{m}}\}} \sum_{\bar{m}=1}^{MN} \left|y_{\pi_{\bar{m}}} - \hat{v}_{\bar{m}}\right|^2 \qquad (7)$$

We consider finding an M-sized subset that covers all the partitions and also has the lowest cost. To accomplish the mentioned task, we build $\boldsymbol{U}$ such that:

$$\boldsymbol{U}_{k,l} \triangleq \|\boldsymbol{y}_k - \hat{\boldsymbol{v}}_l\|_2^2 \text{ for } k, l = 1,2,\cdots,M.$$

The minimization problem for finding the optimal permutation matrix $\boldsymbol{P}_{opt}$ can be recast as,

$$\boldsymbol{P}_{opt} = \arg\min_{\boldsymbol{P}} [\mathbb{1}^T(\boldsymbol{P}\odot\boldsymbol{U})\mathbb{1}] \qquad (8)$$

The problem is in fact an *Assignment Problem* that can be solved efficiently using the Hungarian Algorithm with an O(M^2) computational cost.

Observation 2:

$$\chi_1 \subset \chi_2 \subset \chi_3 \subset \cdots$$

We can always use the appropriate values of $\boldsymbol{v}$ obtained for a smaller s to search for an updated $\boldsymbol{v}$ as we increase s.

5. Final Algorithm

Step 0: Set $s = 1$

Step 1: Monotonically decrease the objective of (3) via cyclic minimization until convergence using (5) and (8)

Step 2: $s \leftarrow s + 1$

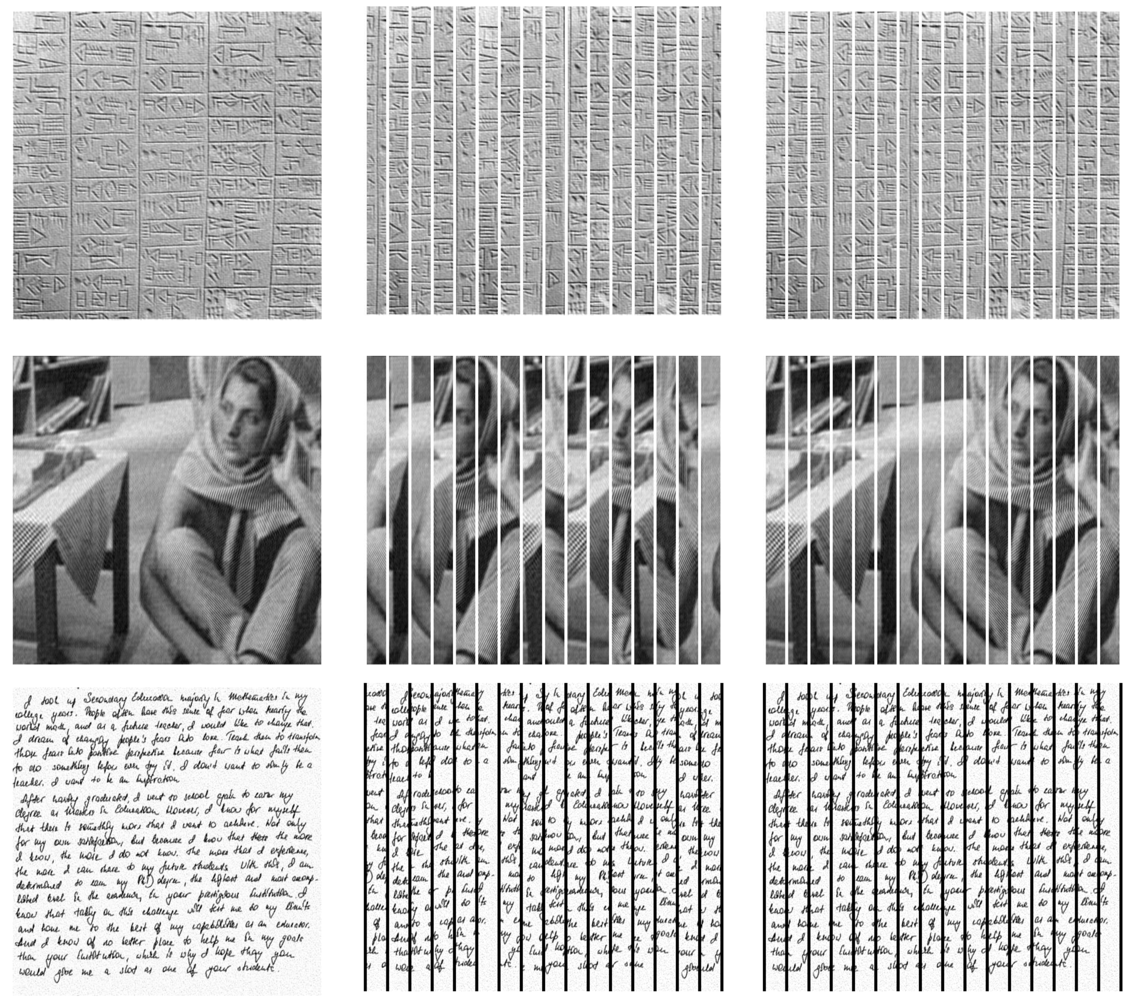
Step 3: Repeat Step 1 until the decrease in the objective of (3) is negligible.

6. Extensions to Two-Dimensional Case

$$\min_{\boldsymbol{P},\bar{\boldsymbol{V}}} \left\|(\boldsymbol{P}\otimes\boldsymbol{I}_N)\bar{\boldsymbol{Y}} - \boldsymbol{\Psi}_C^H\bar{\boldsymbol{V}}\boldsymbol{\Psi}_R^H\right\|_2$$

7. Results

The proposed approach has been tested on several two-dimensional image signals which are known to be sparse in DFT domain. We have used gray scale images of size 512×512 for this purpose. The shredded instances are generated by virtually cutting the document pages vertically into 16 shreds producing 512×32 strips.



(a) (b) (c)

Fig. 2. Reconstruction results: (a) original images, (b) scrambled shredded strips, (c) reconstructed images.

8. Conclusion

- A novel non-convex approach to find the best matching of strip-shredded document.
- The approach is based on the enhancement of sparsity of the observed signal.
- The algorithm was tested on several shredded document pages and images.
- The results obtained suggest that the proposed algorithm demonstrates a great efficiency in terms of the reconstruction rate and computational time.

9. Future Works

As a future research avenue, it would be of great interest to use a relatively large number of partitions for a sufficiently large piece of image that may appear in different orientations, as well as, partitions with crosscut shreds and also with incomplete set of shreds.