Generalized Cyclic Algorithms for Designing Unimodular Sequence Sets with Good (Complementary) Correlation Properties

Arindam Bose
Israel A. Arriaga-Trejoy
Aldo G. Orozco-Lugoz
Mojtaba Soltanalian

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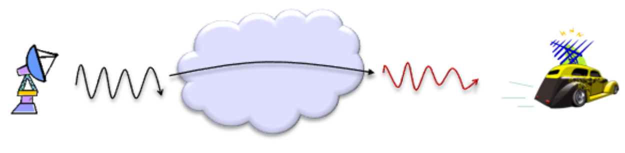
Motivation

Unimodular sequences with good correlation properties

- Unimodular sequences whose auto-correlation function is zero except at some or all correlation lags are of great interest in engineering and technology.
- Where?
    - Channel estimation,
    - System identification.
    - Active sensing,
    - Medical imaging,
    - Radar waveform design and many more...



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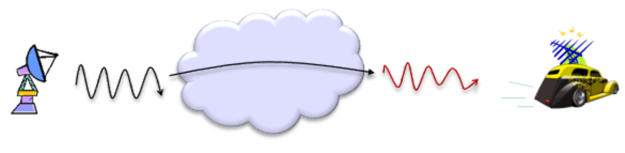
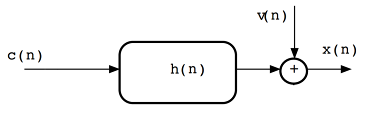


System Identification

Starting with Strictly Linear (SL) Systems



$$x(n) = (h \otimes c)(n) + v(n)$$

- Goal: To estimate the coefficients of the filter $\{h(n)_{n=0}^{L-1}\}$.
- Given: The observation $x(n)$ against input sequence $c(n)$.
- Also, $E\{v(n)\} = 0$, $E\{v(n)v*(m)\} = \sigma_v^2 \delta(m-n)$.

Estimation of SL systems

- Consider a sequence $c(n)$, used to excite the system is chosen to be periodic with period P and mean average power $\sigma_c^2 = P^{-1}\sum_{l=0}^{P-1} |c(l)|^2$.
- The process: $x(n) = \sum_{l=0}^{P-1} h(l)c(n-l) + v(n)$.
- First order statistics: $E\{x(nP+l)\} = \sum_{m=0}^{P-1} h(m)c(l-m)$.
- Estimate of the first order statistics: $\hat{E}\{x(l)\} = \frac{1}{N_P}\sum_{n=0}^{N_P-1} x(nP+l)$, where $N = N_P P$ and N_P is number of periods in the probing sequence for $l = \{0, 1, \cdots, P-1\}$.

Estimation of SL systems

- Define: $\hat{E}\{\boldsymbol{x}\} = [\hat{E}\{x(0)\ \hat{E}\{x(1)\ \cdots \hat{E}\{x(P-1)\}]^T$.
- The estimation: $$\hat{\boldsymbol{h}} = \boldsymbol{C}^{-1}\hat{E}\{\boldsymbol{x}\}$$
- $\boldsymbol{C}$ is a circulant matrix,

$$\boldsymbol{C} = \begin{bmatrix} c(0) & c(P-1) & \cdots & c(2) & c(1) \\ c(1) & c(0) & \cdots & c(3) & c(2) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ c(P-1) & c(P-2) & \cdots & c(1) & c(0) \end{bmatrix}_{P\times P}.$$

- Estimation is straight-forward, easier with structure.

Designing sequences for SL Systems

- Considerable amount of literature is available to design sequences for SL signal processing.
- Formulation is based on minimizing the contribution of the aperiodic or periodic autocorrelation functions for the out-of-phase coefficients.
- Many numerical algorithm has been proposed minimizing metrics such as PSL (peak sideloab level), ISL (integrated sideloab level), PAPR (peak to average power ratio) etc.

However,

These sequences does not show promising efficiency for certain systems. It has been shown in the literature[1], that improved results can be obtained if the full second order characteristics of processes are considered, by performing widely linear (WL) signal processing.

[1] B. Picinbono et al. *Widely linear estimation with complex data.* 1995.

Why WL systems?

- There is an additional degree of freedom that can be exploited when compared to SL systems, which is the complex conjugate of the input signal.
- Modeling of systems using WL structures is quite common in wireless systems, when non-linear radio frequency (RF) impairments such as in-phase and quadrature-phase (I/Q) imbalances are considered in the analysis of signal propagation[2].

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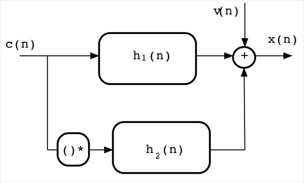
    - In order to compensate the I/Q imbalances at the receiver, it is necessary to compute the complete second order statistics of the received signal, namely the auto-correlation and complementary correlation.

[2] I. A. Arriaga-Trejo et al. *Unimodular Sequences with Low Complementary Autocorrelation Properties.* 2018.

What are Widely Linear (WL) Systems?

- In general, a WL system is characterized by the impulse responses $\{h_1(n)_{n=0}^{L-1}\}$ and $\{h_2(n)_{n=0}^{L-1}\}$.



$$x(n) = (h_1 \otimes c)(n) + (h_2 \otimes c^*)(n) + v(n)$$

- Notice that, x is not a linear function of c, however the k-th order moment of x is completely defined from the k-th order moments of c and c^*.

Estimation of WL systems

- There are $2L$ unknowns, hence $c(n)$ must be chosen with period at least $P = 2L$ and consider each filter h_1 and h_2 with $P/2$ coefficients .
- The process: $x(n) = \sum_{l=0}^{P/2-1} h_1(l)c(n-l) + \sum_{l=0}^{P/2-1} h_2(l)c^*(n-l) + v(n)$.
- First order statistics: $E\{x(nP+l)\} = \sum_{m=0}^{P/2-1} h_1(m)c(l-m) + \sum_{m=0}^{P/2-1} h_2(m)c^*(l-m)$.
- Estimate of the first order statistics: $\hat{E}\{x(l)\} = \frac{1}{N_P}\sum_{n=0}^{N_P-1} x(nP+l)$, where $N = N_P P$ and N_P is number of periods in the probing sequence.

Estimation of WL systems

Define: $E\{\boldsymbol{x}\} = [E\{x(nP)\ E\{x(nP+l)\ \cdots E\{x(nP+P-1)\}]^T$.

$$E\{\boldsymbol{x}\} = \bar{\boldsymbol{C}}_{P/2}\boldsymbol{h}_1 + \bar{\boldsymbol{C}}^*_{P/2}\boldsymbol{h}_2 = \bar{\boldsymbol{C}}\bar{\boldsymbol{h}}$$

where,

$$\begin{aligned}
\boldsymbol{h}_1 &= [h_1(0)\ h_1(1)\ \cdots\ h_1(P/2-1)]^T \\
\boldsymbol{h}_2 &= [h_2(0)\ h_2(1)\ \cdots\ h_2(P/2-1)]^T \\
\bar{\boldsymbol{h}} &= [\boldsymbol{h}_1^T\ \boldsymbol{h}_2^T]^T \\
\bar{\boldsymbol{C}}_{P/2} &= \begin{bmatrix} c(0) & c(P-1) & \cdots & c(\frac{P}{2}+2) & c(\frac{P}{2}+1) \\ c(1) & c(0) & \cdots & c(\frac{P}{2}+3) & c(\frac{P}{2}+2) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ c(P-1) & c(P-2) & \cdots & c(\frac{P}{2}+1) & c(\frac{P}{2}) \end{bmatrix}_{P\times\frac{P}{2}} \\
\bar{\boldsymbol{C}} &= [\bar{\boldsymbol{C}}_{P/2}\ \bar{\boldsymbol{C}}^*_{P/2}].
\end{aligned}$$

Estimation of WL systems

- The estimation: $$\hat{\boldsymbol{h}} = \bar{\boldsymbol{C}}^{-1}\hat{E}\{\boldsymbol{x}\}$$
- Even though the estimator has the same form as SL system identification, notice that $\bar{\boldsymbol{C}}$ is an augmented matrix, instead of a circulant structure.
- In order for $\bar{\boldsymbol{C}}$ to be invertible, $c(n)$ must be complex.
- Also, a WL system cannot be identified using the delta function $\delta(n)$.
- So, they need to be handled differently.

Our goal

Construction of sets of unimodular sequences possessing good correlation as well as good complementary correlation properties.

Problem Formulation

Lets start with

- $\{x_m(n)\}_{n=0,m=1}^{N-1,M}$ as the set of M complex unimodular sequences, each of length N.
- $x_m(n) = e^{j\phi_m(n)}$ for all m, n where the phases $\{\phi_m(n)\}$ can have arbitrary values from $[-\pi, \pi]$.

Definitions

- Cross-correlation:
  $r_{m_1 m_2}(n) = \sum_{k=n}^{N-1} x_{m_1}(k) x_{m_2}^*(k-n) = r_{m_2 m_1}^*(-n)$.
- Complementary cross-correlation (or just relation):
  $\gamma_{m_1 m_2}(n) = \sum_{k=n}^{N-1} x_{m_1}(k) x_{m_2}(k-n) = \gamma_{m_2 m_1}(-n)$.

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- In the case of SL, minimize the integrated sidelobe level (ISL):

$$\mathcal{E}_S \triangleq \sum_{m=1}^{M} \sum_{\substack{n=-N+1 \\ n \neq 0}}^{N-1} |r_{mm}(n)|^2 + \sum_{m_1=1}^{M} \sum_{\substack{m_2=1 \\ m_2 \neq m_1}}^{M} \sum_{n=-(N-1)}^{N-1} |r_{m_1 m_2}(n)|^2.$$

Generalized Weighted ISL for WL

$$
\begin{aligned}
\mathcal{E} \triangleq & \sum_{m=1}^{M} \sum_{\substack{n=-N+1 \\ n\neq 0}}^{N-1} \alpha_n^2 |r_{mm}(n)|^2 \\
& + \sum_{m=1}^{M} \sum_{n=-N+1}^{N-1} \beta_n^2 |\gamma_{mm}(n)|^2 \\
& + \sum_{m_1=1}^{M} \sum_{\substack{m_2=1 \\ m_2\neq m_1}}^{M} \sum_{n=-(N-1)}^{N-1} \alpha_n^2 |r_{m_1 m_2}(n)|^2 \\
& + \sum_{m_1=1}^{M} \sum_{\substack{m_2=1 \\ m_2\neq m_1}}^{M} \sum_{n=-(N-1)}^{N-1} \beta_n^2 |\gamma_{m_1 m_2}(n)|^2
\end{aligned}
$$

By the way,

$$\boldsymbol{R}_n = \begin{bmatrix} r_{11}(n) & r_{12}(n) & \cdots & r_{1M}(n) \\ r_{21}(n) & r_{22}(n) & \cdots & r_{2M}(n) \\ \vdots & \vdots & \ddots & \vdots \\ r_{M1}(n) & r_{M2}(n) & \cdots & r_{MM}(n) \end{bmatrix}_{M\times M}, \quad (1)$$

$$\boldsymbol{\Gamma}_n = \begin{bmatrix} \gamma_{11}(n) & \gamma_{12}(n) & \cdots & \gamma_{1M}(n) \\ \gamma_{21}(n) & \gamma_{22}(n) & \cdots & \gamma_{2M}(n) \\ \vdots & \vdots & \ddots & \vdots \\ \gamma_{M1}(n) & \gamma_{M2}(n) & \cdots & \gamma_{MM}(n) \end{bmatrix}_{M\times M}, \quad (2)$$

where $n = -(N-1), \cdots, 0, \cdots, N-1$.

The Algorithms

The matrix form

$$\begin{aligned}\mathcal{E} &= \alpha_0^2\|\boldsymbol{R}_0 - N\boldsymbol{I}_M\|_F^2 + \beta_0^2\|\boldsymbol{\Gamma}_0\|_F^2 \\ &\quad + 2\sum_{n=1}^{N-1} \alpha_n^2\|\boldsymbol{R}_n\|_F^2 + \beta_n^2\|\boldsymbol{\Gamma}_n\|_F^2 \\ &= \sum_{n=-(N-1)}^{N-1} \alpha_n^2\|\boldsymbol{R}_n - N\boldsymbol{I}_M\delta_n\|_F^2 + \sum_{n=-(N-1)}^{N-1} \beta_n^2\|\boldsymbol{\Gamma}_n\|_F^2.\end{aligned}$$

Parseval-type equality:

$$\mathcal{E} = \frac{1}{2N}\sum_{p=1}^{2N} \|\boldsymbol{\Phi}_r(\omega_p) - \alpha_0 N \boldsymbol{I}_M\|_F^2 + \|\boldsymbol{\Phi}_\gamma(\omega_p)\|_F^2$$

$$\boldsymbol{\Phi}_r(\omega) = \sum_{n=-(N-1)}^{N-1} \alpha_n \boldsymbol{R}_n e^{-jn\omega},$$

$$\boldsymbol{\Phi}_\gamma(\omega) = \sum_{n=-(N-1)}^{N-1} \beta_n \boldsymbol{\Gamma}_n e^{-jn\omega},$$

and $\omega_p = \frac{2\pi}{2N} p$ for $p = 1, \cdots, 2N$.

Consequently,

$$\begin{aligned}\boldsymbol{\Phi}_r(\omega_p) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} A(\omega_p-\psi)\ \boldsymbol{\chi}(\psi)\boldsymbol{\chi}^H(\omega)\ d\psi \\ &= \tilde{\boldsymbol{\chi}}^T(\omega_p)\boldsymbol{A}\tilde{\boldsymbol{\chi}}^*(\omega_p) = (\tilde{\boldsymbol{\chi}}^H(\omega_p)\boldsymbol{A}\tilde{\boldsymbol{\chi}}(\omega_p))^T \\ \boldsymbol{\Phi}_\gamma(\omega_p) &= \tilde{\boldsymbol{\chi}}^T(\omega_p)\boldsymbol{B}\tilde{\boldsymbol{\chi}}(\omega_p).\end{aligned}$$

where, $\tilde{\boldsymbol{\chi}}^T(\omega_p) = [\tilde{\boldsymbol{x}}(0)e^{-j0\omega_p}\ \cdots\ \tilde{\boldsymbol{x}}(N-1)e^{-j(N-1)\omega_p}]^T$
and, $\tilde{\boldsymbol{x}}(n) = [x_1(n)\ x_2(n)\ \cdots\ x_M(n)]^T$.

The reduced criterion

$$\mathcal{E} = \frac{1}{2N}\sum_{p=1}^{2N} \|\tilde{\chi_p}^H\boldsymbol{A}\tilde{\chi_p} - \alpha_0 N\boldsymbol{I}_M\|_F^2 + \|\tilde{\chi_p}^T\boldsymbol{B}\tilde{\chi_p}\|_F^2.$$

Going down from quartic to quadratic:

$$
\begin{aligned}
\mathcal{E} &= \frac{1}{2N}\sum_{p=1}^{2N} \mathrm{tr}\left[(\tilde{\chi_p}^H \boldsymbol{A}\tilde{\chi_p} - \alpha_0 N \boldsymbol{I}_M)^H \right. \\
&\quad \left. \times(\tilde{\chi_p}^H \boldsymbol{A}\tilde{\chi_p} - \alpha_0 N \boldsymbol{I}_M)\right] + \mathrm{tr}\Big([(\tilde{\chi_p}^T \boldsymbol{B}\tilde{\chi_p})^H (\tilde{\chi_p}^T \boldsymbol{B}\tilde{\chi_p})]\Big) \\
&\leq \frac{1}{2N}\sum_{p=1}^{2N} \|\boldsymbol{A}\|_F^2 \|\tilde{\chi_p}\|_F^4 - 2\alpha_0 N \|\boldsymbol{A}\|_F \|\tilde{\chi_p}\|_F^2 \\
&\qquad\qquad + \ \alpha_0^2 N^2 M + \|\boldsymbol{B}\|_F^2 \|\tilde{\chi_p}\|_F^4 \\
&= \frac{\|\boldsymbol{A}\|_F^2 + \|\boldsymbol{B}\|_F^2}{2N} \times \\
&\qquad \sum_{p=1}^{2N} \left(\|\tilde{\chi_p}\|_F^2 - \frac{\alpha_0 N \|\boldsymbol{A}\|_F}{\|\boldsymbol{A}\|_F^2 + \|\boldsymbol{B}\|_F^2} \right)^2 + \text{const.}
\end{aligned}
$$

The reduced optimization problem

$$\min_{\tilde{\chi}_p, \boldsymbol{V}_p} \quad \sum_{p=1}^{2N} \|\tilde{\chi}_p - \boldsymbol{V}_p\|_F^2$$

$$\text{s.t.} \quad |x_m(n)| = 1, \ \|\boldsymbol{V}_p\|_F^2 = \kappa$$

where $\kappa = \dfrac{\alpha_0 N \|\boldsymbol{A}\|_F}{\|\boldsymbol{A}\|_F^2 + \|\boldsymbol{B}\|_F^2}$ which can be solved in a cyclic way.

Note

Both criterions are "almost equivalent" to each other in the sense that if one takes on a small value, so does the other; particularly, the quadratic terms become zero if the above is zero, and vice versa.

Redefine the problem: G-WeCAN

$$\begin{aligned}
\boldsymbol{f}_p &= [e^{-j\omega_p} \ \cdots \ e^{-j2N\omega_p}]^T, \\
\boldsymbol{F} &= [\boldsymbol{f}_1 \ \cdots \ \boldsymbol{f}_{2N}], \\
\bar{\boldsymbol{X}} &= [\boldsymbol{X} \ \boldsymbol{0}]^T_{M\times 2N}, \\
\boldsymbol{V} &= [\boldsymbol{v}_1 \ \cdots \ \boldsymbol{v}_{2N}]^T
\end{aligned}$$

where $\boldsymbol{v}_i$ is the ith column of $\boldsymbol{V}_p$.

$$\min_{\substack{\{x_m(n)\}_{n=0,m=1}^{N-1,M}, \\ \{\boldsymbol{v}_p\}_{p=1}^{2N}}} \left\| \boldsymbol{F}^H \bar{\boldsymbol{X}} - \boldsymbol{V} \right\|_F^2$$

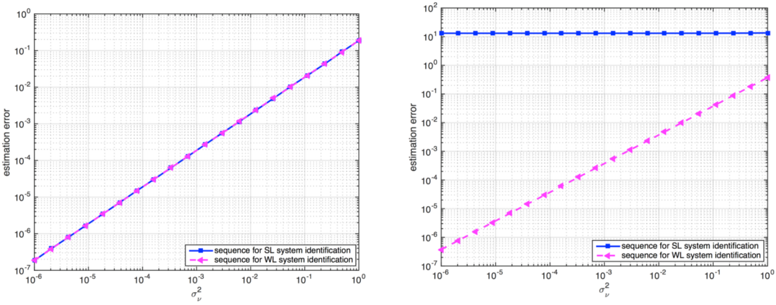
$$\text{s.t.} \quad |x_m(n)| = 1, \ \|\boldsymbol{V}_p\|_F^2 = \kappa.$$

Can be solved using cyclic algorithms.

Numerical Examples

Experimental setup - I

Consider sequence length $N = 128$ is employed to estimate a SL and WL system with $L = 24$.

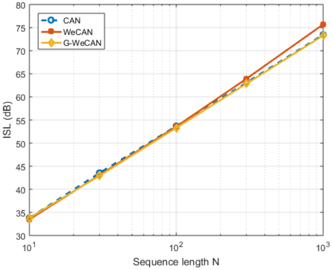


(a) (b)

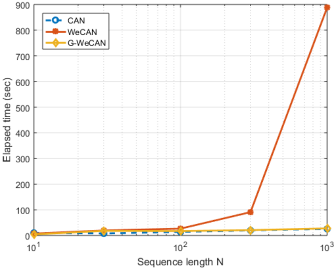
Figure: (a) Variance of the estimation error obtained when identifying the impulse response $\{h(n)\}_{n=0}^{23}$ of a SL system and (b) Variance of estimation error when identifying the responses $\{h_1(n)\}_{n=0}^{23}$ and $\{h_2(n)\}_{n=0}^{23}$ of a widely linear system.

Experimental setup - II

We generate sets of sequences with sequence length $N = \{10, 30, 100, 300, 1000\}$ and $M = 3$ for G-WeCAN to compare with CAN, WeCAN in terms of overall ISL metric.



(a)



(b)

Figure: Comparison of (a) ISL metric and (b) computation times for CAN, WeCAN and G-WeCAN sequence with $N = \{10, 30, 100, 300, 1000\}$ and $M = 3$.

Summary

Pros

- Cyclic algorithms to generate sets of unimodular sequences with good correlation and complementary correlation properties which is important for WL signal processing.
- Can be used to design very long sequences ($N \sim 10^5$) in a short period of time.

Con

- The matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ give similar shapes to the correlation and complementary correlation of the final sequence sets. The algorithm fails when we use different specifications for $\boldsymbol{A}$ and $\boldsymbol{B}$ (future work).

Thank you
and
Questions?