EFFICIENT CONSTRUCTION OF POLYPHASE SEQUENCES WITH OPTIMAL PEAK SIDELOBE LEVEL GROWTH

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ABSTRACT

Sequences with *good correlation* properties form an integral part of many active sensing and communication systems. Although polyphase sequence sets with optimal peak sidelobe level (PSL) growth have been known for years, it has been a long-standing problem to construct polyphase sequences with small PSL growth. In this paper, we introduce a new polynomial-time construction approach to design polyphase sequences with optimal PSL growth from sequence sets with small cross-correlation. The proposed construction approach is based on the observation that if the PSL of the sequence set grows optimally, the constructed polyphase sequence will also enjoy an optimal correlation sidelobe growth. Several numerical examples are provided to illustrate the performance of the construction method.

Index Terms— Correlation, peak sidelobe level (PSL), polyphase sequences, sequence design.

1. INTRODUCTION AND PROBLEM FORMULATION

Sequences with small correlation are often required in many communication systems, range compression and active sensing applications [1, 2]. There is an extensive literature available on designing families of sequence sets with small correlation properties [3–7]. However, there is very little progress on the analytical design of sequences (and not sets)— a problem that is deemed to be difficult from a computational viewpoint [8]. In particular, it is regarded as computationally impractical to perform exhaustive search over a set of polyphase sequences with very large cardinality. In this paper, we propose a polynomial-time efficient method of constructing polyphase sequences with optimal PSL growth from a known sequence sets with small correlation property.

The k-th aperiodic auto-correlation for any sequence $\boldsymbol{x} = [x_1\ x_2\ \cdots\ x_N]^T$ is given as

$$c_{\boldsymbol{x}}(k) = \sum_{n=1}^{N-k} x_n x_{n+k}^*, \quad k \in \{0, \cdots, N-1\}, \tag{1}$$

where $(\cdot)^*$ denotes the complex conjugate for scalers and the conjugate transpose for vectors and matrices. Binary sequences with $x_n = \pm 1$ having low sidelobes have been studied in the literature extensively (known as Barker sequences [9]), with $|c_{\boldsymbol{x}}(k)| \leq 1$, and are known for a few finite lengths. The Barker condition can be extended to the sequences where the entries are of the form $x_n = e^{2\pi i N_n / K}$ where $i^2 = -1$, drawn from a fixed alphabet set of cardinality K. The more general setting allows for polyphase sequences, where the entries of $\boldsymbol{x}$ lie on the unit circle in the complex plane.

Let X be a set of M sequences denoted as $\{x_m(n)\}_{m,n=1}^{M,N}$ each of which is of length N and constrained to have the same finite energy, i.e.

$$\sum_{n=1}^{N} |x_m(n)|^2 = N, \qquad m \in \{1, \cdots, M\}. \tag{2}$$

From this point onward we represent $\{x_m(n)\}_{n=1}^{N}$ simply by its vector notation $\boldsymbol{x}_m$. The aperiodic cross-correlations of the sequences $\boldsymbol{x}_p$ and $\boldsymbol{x}_q$ from the set X at shift k are defined as

$$\begin{aligned} r_{X;pq}(k) &\triangleq \sum_{l=k+1}^{N} x_p(l) x_q^*(l-k) = r_{pq}^*(-k) \\ p, q &\in \{1, \cdots, M\}, \qquad k \in \{0, \cdots, N-1\}, \end{aligned} \tag{3}$$

The aperiodic auto-correlation of $\boldsymbol{x}_m \in X$ can be obtained from (3) by considering $p = q = m$. In this context, we construct the correlation matrix $\boldsymbol{R}_{X;k}$ for k-th lag using $r_{X;pq}(k)$ such that,

$$\boldsymbol{R}_{X;k} = \begin{bmatrix} r_{X;11}(k) & r_{X;21}(k) & \dots & r_{X;M1}(k) \\ r_{X;12}(k) & r_{X;22}(k) & \dots & r_{X;M2}(k) \\ \vdots & \vdots & \ddots & \vdots \\ r_{X;1M}(k) & r_{X;2M}(k) & \dots & r_{X;MM}(k) \end{bmatrix}$$

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$$k = -N+1, \cdots, 0, \cdots, N-1. \quad (4)$$

Using the "shift matrix",

$$\boldsymbol{J}_k = \begin{bmatrix} \overbrace{0 \cdots 1}^{k+1} & & 0 \\ & \ddots & \\ & & 1 \\ 0 & & \end{bmatrix}^T_{N \times N} = \boldsymbol{J}_{-k}^T, \qquad k = 0, \cdots, N-1 \quad (5)$$

we can rewrite $\boldsymbol{R}_{X;k}$ as

$$\boldsymbol{R}_{X;k} = X^* \boldsymbol{J}_k X = \boldsymbol{R}^*_{X;-k}, \qquad k = 0, \cdots, N-1 \quad (6)$$

There exists a rich literature [7, 10, 11] on generating sequence sets that asymptotically meet *periodic* correlation bounds. However, the aperiodic case is considered to be more difficult, though interesting. In the sequel, we address the well established problem of designing methods of polyphase sequence sets whose out-of-phase (i.e. with $k \neq 0$) aperiodic auto-correlations lags are, collectively, small. One of the several important measures of "smallness" is the *Peak Sidelobe Level* (PSL) of the sequence set X defined as,

$$\begin{aligned} &\mathcal{P}_X \triangleq \max_{p,q} \{|r_{X;pq}(k)|\}, \\ &p, q \in \{1, \cdots, M\} \\ &\text{and } k \in 0, \cdots, N-1 \quad (k \neq 0 \text{ if } p = q) \end{aligned} \quad (7)$$

which is the most relevant to our analysis.

The Welch bounds [12] are the most well-known theoretical limits on the collective smallness of correlation values of the sequence sets. The Welch lower bound of PSL is given as,

$$\mathcal{P}_X \geq N \sqrt{\frac{M-1}{2NM - M - 1}} \triangleq B_{\mathcal{P}_X}. \quad (8)$$

We assume that $2 \leq M \ll N$ and that M behaves as $\mathcal{O}(1)$ with respect to the sequence length N. Note that, the above lower bound can be achieved conveniently by the well-known analytically-built sequence sets, or by using computational design frameworks such as CAN introduced in [4]. Keeping that in mind, from (8) it is not difficult to observe that for an X meeting the Welch bound, and as N grows large,

$$\mathcal{P}_X \lessapprox \frac{1}{\sqrt{2}} \sqrt{N}, \quad (9)$$

which implies that the PSL of the sequence set X behaves as $\mathcal{O}(\sqrt{N})$ as $N \to \infty$ [12].

2. PROPOSED METHOD OF CONSTRUCTION

Let $\boldsymbol{s}$ be a polyphase sequence with entries $\{s(n)\}_{n=1}^N$. We design $\boldsymbol{s}$ as a linear combination of the sequence set $\boldsymbol{X}$ (represented as a matrix) and a weighted multiplier $\boldsymbol{\phi}$, i.e.

$$\boldsymbol{s} = \sum_{m=1}^{M} \phi(m) \boldsymbol{x}_m = \boldsymbol{X}\boldsymbol{\phi} \quad (10)$$

where $\boldsymbol{\phi} \in \mathbb{C}^M$ is a column vector denoted as $\{\phi(m)\}_{m=1}^M$. Note that $\boldsymbol{s}^*\boldsymbol{s} = N$. The aperiodic auto-correlation lags of $\boldsymbol{s}$ at shift k are given as,

$$\begin{aligned} &r_{\boldsymbol{s}}(k) \\ &\triangleq \sum_{l=k+1}^{N} s(l) s^*(l-k) \\ &= \sum_{p=1}^{M} \sum_{q=1}^{M} \left(\phi(p)\phi^*(q) \sum_{l=k+1}^{N} \boldsymbol{x}_p(l) \boldsymbol{x}_q^*(l-k) \right). \end{aligned} \quad (11)$$

Now one can deduce from (11) that,

$$\begin{aligned} |r_{\boldsymbol{s}}(k)| &\leq \sum_{p=1}^{M} \sum_{q=1}^{M} |\phi(p)| \, |\phi^*(q)| \, \left| r_{pq}^{\boldsymbol{X}}(k) \right| \\ &\leq \max_{p,q} \{ |r_{pq}^{\boldsymbol{X}}(k)| \} \left(\sum_{p=1}^{M} \sum_{q=1}^{M} |\phi(p)| |\phi^*(q)| \right) \\ &\leq \mathcal{P}_{\boldsymbol{X}} \, \|\boldsymbol{\phi}\|_1^2. \end{aligned} \quad (12)$$

Combining (9) and (12), we can say that the PSL of polyphase sequence $\boldsymbol{s}$ is bounded as

$$\mathcal{P}_{\boldsymbol{s}} \lessapprox \frac{\|\boldsymbol{\phi}\|_1^2}{\sqrt{2}} \sqrt{N}. \quad (13)$$

Now observe that,

$$[\boldsymbol{X}^*\boldsymbol{X}]_{p,q} = \begin{cases} N & p = q \\ \alpha_{p,q} & p \neq q \end{cases} \qquad p, q \in \{1, 2, \cdots, M\}. \quad (14)$$

In other words, $\boldsymbol{X}^*\boldsymbol{X}$ can be written as a summation of two $M \times M$ matrices, viz.

$$\boldsymbol{X}^*\boldsymbol{X} \triangleq N\boldsymbol{I}_M + \boldsymbol{Q}. \quad (15)$$

Also note that, according to (8), for a sequence set $\boldsymbol{X}$ achieving the Welch bound,

$$|\alpha_{p,q}| \leq N \sqrt{\frac{M-1}{2NM - M - 1}}. \quad (16)$$

Thus, from (15) and (16), we can conclude that,

$$\|\boldsymbol{Q}\|_F \leq N \sqrt{(M^2 - M) \left(\frac{M-1}{2NM - M - 1} \right)}, \quad (17)$$

where $\|\cdot\|_F$ represents the Frobenius norm of the matrix argument.

Let $\boldsymbol{X} = \boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}^*$ represent the Singular Value Decomposition (SVD) of $\boldsymbol{X}$, where $\boldsymbol{U}$ and $\boldsymbol{V}$ are unitary matrices of size $N \times N$ and $M \times M$, respectively, and $\boldsymbol{\Sigma}$ is an $N \times M$ diagonal matrix. $\{\sigma_m\}_{m=1}^{M}$ denote the singular values of $\boldsymbol{X}$. Note that,

$$\boldsymbol{X}^*\boldsymbol{X} = \boldsymbol{V}\boldsymbol{\Sigma}^2\boldsymbol{V}^* \tag{18}$$

As a result, from (15) and (18)

$$|\sigma_m|^2 = \boldsymbol{e}_m^T\boldsymbol{\Sigma}^2\boldsymbol{e}_m = N + \boldsymbol{e}_m^T\boldsymbol{V}^*\boldsymbol{Q}\boldsymbol{V}\boldsymbol{e}_m. \tag{19}$$

where $\boldsymbol{e}_m$ denotes the m-th standard basis vector of $\mathbb{C}^N$. It is clear from (17) that

$$|\boldsymbol{e}_m^T\boldsymbol{V}^H\boldsymbol{Q}\boldsymbol{V}\boldsymbol{e}_m| \leq N\sqrt{\frac{M(M-1)^2}{2NM-M-1}}.$$

Consequently, it can be easily verified that

$$|\sigma_m|^2 \geq N - N\sqrt{\frac{M(M-1)^2}{2NM-M-1}}. \tag{20}$$

Further observe that,

$$\begin{aligned} \|\boldsymbol{X}^+\|_F^2 &= \sum_{m=1}^{M}\frac{1}{|\sigma_m|^2} \\ &\leq \frac{M}{N - N\sqrt{\frac{M(M-1)^2}{2NM-M-1}}} \end{aligned} \tag{21}$$

where $\boldsymbol{X}^+$ is the Moore–Penrose pseudoinverse of $\boldsymbol{X}$, defined as

$$\boldsymbol{X}^+ \triangleq (\boldsymbol{X}^*\boldsymbol{X})^{-1}\boldsymbol{X}^*. \tag{22}$$

Hence, by definition $\boldsymbol{X}\boldsymbol{X}^+$ is Hermitian, and $\boldsymbol{X}^+\boldsymbol{X} = \boldsymbol{I}_M$.

Thus, from (10) we conclude that $\boldsymbol{\phi} = \boldsymbol{X}^+\boldsymbol{s}$, and therefore,

$$\begin{aligned} \|\boldsymbol{\phi}\|_2^2 &\leq \|\boldsymbol{X}^+\|_F^2\|\boldsymbol{s}\|_2^2 \\ &\leq \frac{M}{1 - \sqrt{\frac{M(M-1)^2}{2NM-M-1}}}. \end{aligned} \tag{23}$$

Note that, according to the Cauchy-Schwarz inequality,

$$\left(\sum_{m=1}^{M}|\phi(m)|\right)^2 \leq \left(\sum_{m=1}^{M}|\phi(m)|^2\right)\left(\sum_{m=1}^{M}1^2\right). \tag{24}$$

It follows from the above that

$$\|\boldsymbol{\phi}\|_1 \leq M\sqrt{\frac{1}{1-\sqrt{\frac{M(M-1)^2}{2NM-M-1}}}}. \tag{25}$$

Now, it is not hard to verify that the upper bound above converges to M as $N \to \infty$ showing that μ behaves as $\mathcal{O}(1)$ with respect to the sequence length N, as N grows large. Hence from (13) we can conclude that $\mathcal{P}_{\boldsymbol{s}}$ behaves like $\mathcal{O}(\sqrt{N})$.

Considering $\boldsymbol{X}$ as a basis with good correlation properties, we can think of designing $\{s(n)\}_{n=1}^{N}$ by solving the following minimization criterion:

$$\begin{aligned} \min_{\{s(n)\}_{n=1}^{N};\{\phi(m)\}_{m=1}^{M}} \quad & f = \|\boldsymbol{X}\boldsymbol{\phi} - \boldsymbol{s}\|_2^2 \\ \text{s.t.} \quad & \boldsymbol{s} \in \Omega, \end{aligned} \tag{26}$$

where

$$\begin{aligned} \boldsymbol{\phi} &= [\phi(1)\ \phi(2)\ \cdots\ \phi(M)]^T, \\ \boldsymbol{s} &= [s(1)\ s(2)\ \cdots\ s(N)]^T, \end{aligned}$$

Ω is the set of polyphase sequences, and $\boldsymbol{X}$ is a $N \times M$ matrix.

A natural approach to tackle the "nearest-vector" optimization problem in (26) is to use cyclic minimization. Namely, we first fix $\boldsymbol{s}$ and compute the optimum $\widehat{\boldsymbol{\phi}}$ that minimizes f, which can be handled by using the least-squares solution:

$$\widehat{\boldsymbol{\phi}} = \boldsymbol{X}^+\boldsymbol{s}. \tag{27}$$

Next, we fix $\boldsymbol{\phi}$ and the optimum minimizer $\widehat{\boldsymbol{s}}$ of (26) can be obtained as

$$\widehat{\boldsymbol{s}} = \text{SGN}(\boldsymbol{X}\boldsymbol{\phi}) \tag{28}$$

where $\text{SGN}(\cdot)$ is the signum function defined on vector/matrix arguments. On the other hand, one can also tackle (26) directly (not in a cyclic manner) by substituting (27) in the original objective function, leading to:

$$\min_{\{s(n)\}_{n=1}^{N}} \|\boldsymbol{X}\boldsymbol{X}^+\boldsymbol{s} - \boldsymbol{s}\|_2^2 \tag{29}$$

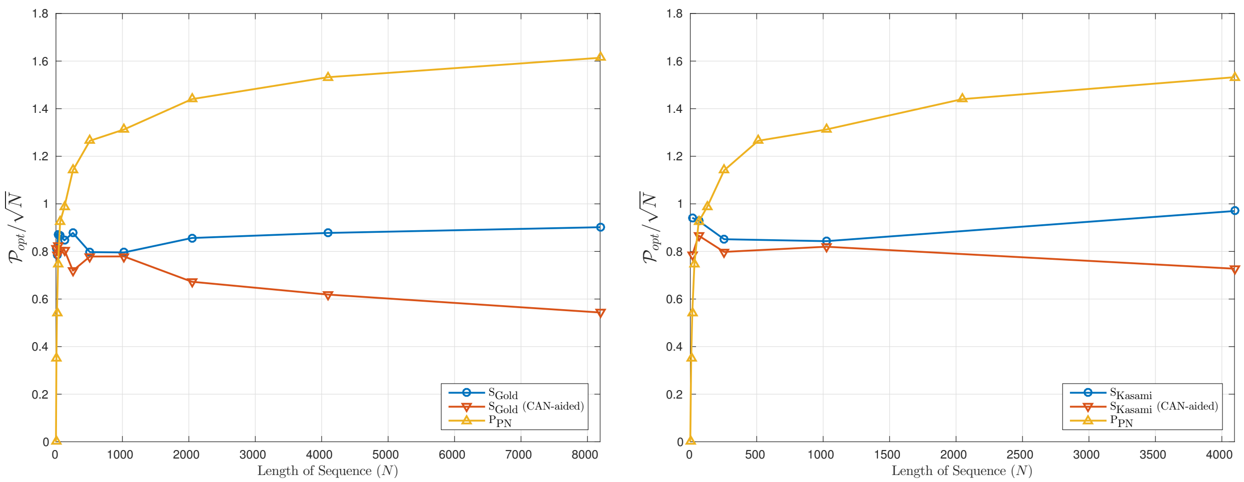
The objective function in (29) can be simplified as,

$$\begin{aligned} &\|\boldsymbol{X}\boldsymbol{X}^+\boldsymbol{s} - \boldsymbol{s}\|_2^2 \\ &= (\boldsymbol{X}\boldsymbol{X}^+\boldsymbol{s} - \boldsymbol{s})^*(\boldsymbol{X}\boldsymbol{X}^+\boldsymbol{s} - \boldsymbol{s}) \\ &= \boldsymbol{s}^*\boldsymbol{X}\boldsymbol{X}^+\boldsymbol{X}\boldsymbol{X}^+\boldsymbol{s} \ + \ \boldsymbol{s}^*\boldsymbol{s} \ - \ 2\boldsymbol{s}^*\boldsymbol{X}\boldsymbol{X}^+\boldsymbol{s} \\ &= \ -\ \boldsymbol{s}^*\boldsymbol{X}\boldsymbol{X}^+\boldsymbol{s} \ + \ N. \end{aligned} \tag{30}$$

As a result, the minimization problem in (29) can be alternatively described as,

$$\max_{\boldsymbol{s} \in \Omega} \quad \boldsymbol{s}^*\boldsymbol{X}\boldsymbol{X}^+\boldsymbol{s} \tag{31}$$

where the quadratic term $\boldsymbol{X}\boldsymbol{X}^+$ is rank deficient. Note that, in general, the maximization of a full-rank semi definite quadratic form over a polyphase sequence is $\mathcal{NP}$-hard. However, for rank deficient quadratic forms, the optimization problem can be solved with polynomial complexity with respect to the sequence length N as described in [13, 14]. Especially, [14] shows a construction algorithm with a $\mathcal{O}(N^{2M})$ computational cost.



(a) (b)

Fig. 1. The PSL growth of constructed binary sequences vs. length N obtained from different sequence families: (a) Gold sequence and (b) Kasami sequence,

3. NUMERICAL RESULTS

In this section, we present several examples to investigate the performance of our construction approach. We construct new polyphase sequences from the well-known sequence sets with low cross-correlation such as Gold and Kasami sequence families for allowable values of sequence length. We compare the PSL growth of obtained optimal sequences using (31) with a $\sqrt{N}$ growth, where N is the sequence length. Moreover, the CAN algorithm of [4] is used to lower the PSL of the obtained sequences even further. It is worth noting that, the CAN algorithm, by itself, is not able to find polyphase sequences with a similar optimal PSL growth.

Fig. 1 shows the optimal PSL growth $\mathcal{P}_{opt}$ of the obtained sequences. It is evident from the figures that the value $\mathcal{P}_{opt}/\sqrt{N}$ indeed remains near-constant. Moreover, Fig. 1 depicts the PSL growth of randomly generated *Pseudo-Noise* (PN) sequences of same length which, showing an $\mathcal{O}(\sqrt{N \ln N})$ growth.

4. DISCUSSION AND FUTURE WORK

In this paper, we presented a polynomial-time approach to the construction of polyphase sequences. We aimed at generating sequences that has good auto-correlation, and also has an optimal sidelobe growth in an asymptotic sense. This approach can be used to design very long sequences (of length up to $N \sim 2^{12}$)— owing to fast computational approaches such as the CAN algorithm, and particularly, the polynomial-time nature of the proposed approach. Several numerical examples were presented to demonstrate the performance of the construction approach.

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