Generalized Cyclic Algorithms for Designing Unimodular Sequence Sets with Good (Complementary) Correlation Properties

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***Abstract*—Unimodular sequences with good correlation properties have attracted significant research interest due to their applications in several areas such as radar sensing and communications. In this paper, we address the problem of designing unimodular sequence sets with good correlation and complementary correlation properties using generalized cyclic algorithms for the minimization of weighted integrated sidelobe level (WISL) based metrics. The set of sequences are obtained by considering their complete second-order characterization, which has been proven to be beneficial for identification and sensing systems employing widely linear signal processing. Several numerical examples have been presented to illustrate the performance of the proposed algorithms.**

***Index Terms*—Auto-correlation, cyclic algorithms, integrated sidelobe level, unimodular sequences, waveform design.**

I. Introduction

The design of unimodular sequence sets with good correlation properties has been studied extensively in the recent years due to their applications in various fields such as radar [1] and sonar signal design [2], [3], channel estimation, system identification, medical imaging and telecommunications [4], among others. Considerable efforts have been made to generate such sets of sequences using several methods, including alternating projections [5], [6], majorization-minimization techniques [7], [8] and quasi-Newton methods [9], all by minimizing the qualitative metrics based on the peak sidelobe level (PSL) or the integrated sidelobe level (ISL) [10]–[15]. Such recently developed methods rely on Fast Fourier Transform (FFT) operation which makes the design very efficient.

Although the construction of sequences or sequence sets with low sidelobe levels has been broadly addressed in the literature [15]–[27], it should be noted that the aforementioned methods only consider minimizing the contribution of the periodic or aperiodic correlation functions for the out-of-phase ($k \neq 0$) coefficients based on the fact that the designed sequences will only be employed on strictly linear (SL) systems, where only the correlation functions are observed. However, by considering the complete second-order characterization, the performence of the sensing and identification systems can be further enhanced by performing widely linear (WL) signal processing, as shown in [28], [29]. Unlike SL systems, WL systems exploit the complex conjugate of the input signal as an additional degree of freedom for linear processing. Modeling of systems using WL structures is quite common in wireless systems, when non-linear radio frequency (RF) impairments such as in-phase and quadrature-phase (I/Q) imbalances are considered in the analysis of signal propagation [30]–[32].

In this paper, we consider the construction of sets of unimodular sequences possessing good correlation as well as good complementary correlation properties by promoting both desired properties using cyclic algorithms.

Notation: We use bold lowercase letters for vectors and bold uppercase letters for matrices. $(\cdot)^*$, $(\cdot)^T$ and $(\cdot)^H$ denote the complex conjugate, transpose and conjugate transpose of the vector/matrix, respectively. $\|\boldsymbol{x}\|_n$ or the l_n-norm of the vector $\boldsymbol{x}$ is defined as $\{\sum_k |x(k)|^n\}^{\frac{1}{n}}$ where $x(k)$ is the k-th entry of $\boldsymbol{x}$. $\|\boldsymbol{A}\|_F$ or the Frobenius norm of matrix $\boldsymbol{A}$ with entries $\{a_i(j)\}_{i=1,j=1}^{m,n}$ is defined as $\sqrt{\sum_{i=1}^m \sum_{j=1}^n |a_i(j)|^2}$.

II. Problem Formulation

Let $\{x_m(n)\}_{n=0,m=1}^{N-1,M}$ denote the set of M complex unimodular sequences, each of length N, to be designed. We assume that, $x_m(n) = e^{j\phi_m(n)}$ for all m, n and the phases $\{\phi_m(n)\}$ can have arbitrary values from $[-\pi, \pi]$. The aperiodic cross-correlation $r_{m_1 m_2}(n)$ and complementary cross-correlation $\gamma_{m_1 m_2}(n)$ of any member sequences $\{x_{m_1}(k)\}_{k=0}^{N-1}$ and $\{x_{m_2}(k)\}_{k=0}^{N-1}$ at lag n are given as,

$$r_{m_1 m_2}(n) = \sum_{k=n}^{N-1} x_{m_1}(k) x_{m_2}^*(k-n) = r_{m_2 m_1}^*(-n) \tag{1}$$

$$\gamma_{m_1 m_2}(n) = \sum_{k=n}^{N-1} x_{m_1}(k) x_{m_2}(k-n) = \gamma_{m_2 m_1}(-n) \tag{2}$$

for $n = 0, 1, \cdots, N-1$. Note that, the auto-correlation and complementary auto-correlation coefficients can also be derived easily by using $m_1 = m_2 = m$.

In the case of SL signal processing, we design sequences with good correlation properties by minimizing the following criterion based on the ISL of the said sequences:

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$$\mathcal{E}_S \triangleq \sum_{m=1}^{M} \sum_{\substack{n=-N+1 \\ n\neq 0}}^{N-1} |r_{mm}(n)|^2 + \sum_{m_1=1}^{M} \sum_{\substack{m_2=1 \\ m_2\neq m_1}}^{M} \sum_{n=-(N-1)}^{N-1} |r_{m_1 m_2}(n)|^2. \quad (3)$$

To facilitate the discussion in the following, we denote the sequence set in its matrix form, i.e. $\boldsymbol{X} = [\boldsymbol{x}_1 \ \boldsymbol{x}_2 \ \cdots \ \boldsymbol{x}_m \ \cdots \ \boldsymbol{x}_M]_{N\times M}$ where $\boldsymbol{x}_m = [x_m(0) \ x_m(1) \ \cdots \ x_m(N-1)]^T$. Moreover, the covariance and complementary covariance matrices of the sequences are given for different lags as

$$\boldsymbol{R}_n = \begin{bmatrix} r_{11}(n) & r_{12}(n) & \cdots & r_{1M}(n) \\ r_{21}(n) & r_{22}(n) & \cdots & r_{2M}(n) \\ \vdots & \vdots & \ddots & \vdots \\ r_{M1}(n) & r_{M2}(n) & \cdots & r_{MM}(n) \end{bmatrix}_{M\times M}, \quad (4)$$

$$\boldsymbol{\Gamma}_n = \begin{bmatrix} \gamma_{11}(n) & \gamma_{12}(n) & \cdots & \gamma_{1M}(n) \\ \gamma_{21}(n) & \gamma_{22}(n) & \cdots & \gamma_{2M}(n) \\ \vdots & \vdots & \ddots & \vdots \\ \gamma_{M1}(n) & \gamma_{M2}(n) & \cdots & \gamma_{MM}(n) \end{bmatrix}_{M\times M}, \quad (5)$$

where $n = -(N-1), \cdots, 0, \cdots, N-1$. Next, we consider the problem of designing unimodular sequences with good correlation as well as complementary correlation properties by minimizing the following generalized weighted ISL (WISL) criterion,

$$\begin{aligned} \mathcal{E} \triangleq & \sum_{m=1}^{M} \sum_{\substack{n=-N+1 \\ n\neq 0}}^{N-1} \alpha_n^2 |r_{mm}(n)|^2 \quad (6) \\ & + \sum_{m=1}^{M} \sum_{n=-N+1}^{N-1} \beta_n^2 |\gamma_{mm}(n)|^2 \\ & + \sum_{m_1=1}^{M} \sum_{\substack{m_2=1 \\ m_2\neq m_1}}^{M} \sum_{n=-(N-1)}^{N-1} \alpha_n^2 |r_{m_1 m_2}(n)|^2 \\ & + \sum_{m_1=1}^{M} \sum_{\substack{m_2=1 \\ m_2\neq m_1}}^{M} \sum_{n=-(N-1)}^{N-1} \beta_n^2 |\gamma_{m_1 m_2}(n)|^2 \end{aligned}$$

where $\{\alpha_n\}_{n=0}^{N-1}$ and $\{\beta_n\}_{n=0}^{N-1}$ are real-valued weights with $\alpha_n = \alpha_{-n}$ and $\beta_n = \beta_{-n}$. In this paper, we introduce a generalized approach to minimize the criterion in (6) by using the minimization techniques for ISL related metrics such as CAN and WeCAN, introduced in [5], [11].

III. Generalized WeCAN

The Generalized WeCAN (G-WeCAN) algorithm is associated with the criterion $\mathcal{E}$ in (6), which can be written in matrix form as,

$$\begin{aligned} \mathcal{E} &= \alpha_0^2 \|\boldsymbol{R}_0 - N\boldsymbol{I}_M\|_F^2 + \beta_0^2 \|\boldsymbol{\Gamma}_0\|_F^2 \quad (7) \\ & \quad + 2\sum_{n=1}^{N-1} \alpha_n^2 \|\boldsymbol{R}_n\|_F^2 + \beta_n^2 \|\boldsymbol{\Gamma}_n\|_F^2 \\ &= \sum_{n=-(N-1)}^{N-1} \alpha_n^2 \|\boldsymbol{R}_n - N\boldsymbol{I}_M \delta_n\|_F^2 + \sum_{n=-(N-1)}^{N-1} \beta_n^2 \|\boldsymbol{\Gamma}_n\|_F^2 \end{aligned}$$

where δ_n denotes the Kronecker delta. Furthermore, following the proof in [5] for the case of $M = 1$, it can be shown that the criterion in (7) can be equivalently written as a Parseval-type equality:

$$\mathcal{E} = \frac{1}{2N} \sum_{p=1}^{2N} \|\boldsymbol{\Phi}_r(\omega_p) - \alpha_0 N \boldsymbol{I}_M\|_F^2 + \|\boldsymbol{\Phi}_\gamma(\omega_p)\|_F^2 \quad (8)$$

in which,

$$\boldsymbol{\Phi}_r(\omega) = \sum_{n=-(N-1)}^{N-1} \alpha_n \boldsymbol{R}_n e^{-jn\omega}, \quad (9)$$

$$\boldsymbol{\Phi}_\gamma(\omega) = \sum_{n=-(N-1)}^{N-1} \beta_n \boldsymbol{\Gamma}_n e^{-jn\omega}, \quad (10)$$

and $\{\omega_p\}$ are the Fourier frequencies given as, $\omega_p = \frac{2\pi}{2N} p$ for $p = 1, \cdots, 2N$.

Note that, by choosing α_n and β_n appropriately, we can help shape the correlation lags in the desired form. Particularly for convenience, we choose α_0 and β_0 large enough to ensure that the matrices

$$\boldsymbol{A} = \begin{bmatrix} \alpha_0 & \cdots & \alpha_{N-1} \\ \alpha_1 & \cdots & \alpha_{N-2} \\ \vdots & \ddots & \vdots \\ \alpha_{N-1} & \cdots & \alpha_0 \end{bmatrix}, \quad \boldsymbol{B} = \begin{bmatrix} \beta_0 & \cdots & \beta_{N-1} \\ \beta_1 & \cdots & \beta_{N-2} \\ \vdots & \ddots & \vdots \\ \beta_{N-1} & \cdots & \beta_0 \end{bmatrix} \quad (11)$$

become positive semidefinite (i.e. $\boldsymbol{A} \succeq 0$ and $\boldsymbol{B} \succeq 0$).

Now, note that the following discrete (inverse) Fourier transform relations hold:

$$\{\alpha_n \boldsymbol{R}_n\} \xleftrightarrow{\text{(I)DFT}} \boldsymbol{\Phi}_r(\omega) = A(\omega) * (\boldsymbol{\chi}(\omega)\boldsymbol{\chi}^H(\omega)) \quad (12)$$

$$\{\beta_n \boldsymbol{\Gamma}_n\} \xleftrightarrow{\text{(I)DFT}} \boldsymbol{\Phi}_\gamma(\omega) = B(\omega) * (\boldsymbol{\chi}(\omega)\boldsymbol{\chi}^T(\omega)) \quad (13)$$

where,

$$\boldsymbol{\chi}(\omega) = \sum_{n=0}^{N-1} \tilde{\boldsymbol{x}}(n) e^{-jn\omega}, \quad (14)$$

$$A(\omega) = \sum_{k=-(N-1)}^{N-1} \alpha_k e^{-jk\omega}, \quad (15)$$

$$B(\omega) = \sum_{k=-(N-1)}^{N-1} \beta_k e^{-jk\omega}, \quad (16)$$

and $\tilde{\boldsymbol{x}}(n) = [x_1(n)\ x_2(n)\ \cdots\ x_M(n)]^T$. Consequently, one can verify that,

$$\begin{aligned}
\boldsymbol{\Phi}_r(\omega_p) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} A(\omega_p-\psi)\ \boldsymbol{\chi}(\psi)\boldsymbol{\chi}^H(\omega)\ d\psi \qquad (17)\\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} \sum_{k=-(N-1)}^{N-1} \alpha_k e^{-jk(\omega_p-\psi)}\boldsymbol{\chi}(\psi)\boldsymbol{\chi}^H(\psi)\ d\psi\\
&= \sum_{k=-(N-1)}^{N-1}\sum_{n=0}^{N-1}\sum_{\tilde{n}=0}^{N-1} \alpha_k \tilde{\boldsymbol{x}}(n)\tilde{\boldsymbol{x}}^*(\tilde{n})\ e^{-jk\omega_p}\\
&\qquad\times\ \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{j(k-n+\tilde{n})\psi}\ d\psi\\
&= \sum_{n=0}^{N-1}\sum_{\tilde{n}=0}^{N-1} \alpha_{n-\tilde{n}}\tilde{\boldsymbol{x}}(n)\tilde{\boldsymbol{x}}^*(\tilde{n})e^{-j(n-\tilde{n})\omega_p}\\
&= \tilde{\boldsymbol{\chi}}^T(\omega_p)\boldsymbol{A}\tilde{\boldsymbol{\chi}}^*(\omega_p) = (\tilde{\boldsymbol{\chi}}^H(\omega_p)\boldsymbol{A}\tilde{\boldsymbol{\chi}}(\omega_p))^T
\end{aligned}$$

where, $\tilde{\boldsymbol{\chi}}(\omega_p) = [\tilde{\boldsymbol{x}}(0)e^{-j0\omega_p}\ \cdots\ \tilde{\boldsymbol{x}}(N-1)e^{-j(N-1)\omega_p}]^T$. Similarly for $\boldsymbol{\Phi}_\gamma(\omega_p)$, it can be given as

$$\boldsymbol{\Phi}_\gamma(\omega_p) = \tilde{\boldsymbol{\chi}}^T(\omega_p)\boldsymbol{B}\tilde{\boldsymbol{\chi}}(\omega_p). \qquad (18)$$

Therefore, by denoting $\tilde{\boldsymbol{\chi}}(\omega_p)$ simply as $\tilde{\boldsymbol{\chi}_p}$, the criterion in (8) can be rewritten as

$$\mathcal{E} = \frac{1}{2N}\sum_{p=1}^{2N} \|\tilde{\boldsymbol{\chi}_p}^H\boldsymbol{A}\tilde{\boldsymbol{\chi}_p} - \alpha_0 N\boldsymbol{I}_M\|_F^2 + \|\tilde{\boldsymbol{\chi}_p}^T\boldsymbol{B}\tilde{\boldsymbol{\chi}_p}\|_F^2. \qquad (19)$$

Note that the above function is quartic in $\{x_m(n)\}_{n=0,m=1}^{N-1,M}$, which makes the optimization difficult. To facilitate a transformation to a quadratic objective, we simplify (19) as

$$\begin{aligned}
\mathcal{E} &= \frac{1}{2N}\sum_{p=1}^{2N} \mathrm{tr}\left[(\tilde{\boldsymbol{\chi}_p}^H\boldsymbol{A}\tilde{\boldsymbol{\chi}_p} - \alpha_0 N\boldsymbol{I}_M)^H\right.\\
&\quad\left.\times(\tilde{\boldsymbol{\chi}_p}^H\boldsymbol{A}\tilde{\boldsymbol{\chi}_p} - \alpha_0 N\boldsymbol{I}_M)\right] + \mathrm{tr}\big([(\tilde{\boldsymbol{\chi}_p}^T\boldsymbol{B}\tilde{\boldsymbol{\chi}_p})^H(\tilde{\boldsymbol{\chi}_p}^T\boldsymbol{B}\tilde{\boldsymbol{\chi}_p})]\big)\\
&\le \frac{1}{2N}\sum_{p=1}^{2N} \|\boldsymbol{A}\|_F^2\|\tilde{\boldsymbol{\chi}_p}\|_F^4 - 2\alpha_0 N\|\boldsymbol{A}\|_F\|\tilde{\boldsymbol{\chi}_p}\|_F^2\\
&\qquad +\ \alpha_0^2N^2M + \|\boldsymbol{B}\|_F^2\|\tilde{\boldsymbol{\chi}_p}\|_F^4\\
&= \frac{\|\boldsymbol{A}\|_F^2+\|\boldsymbol{B}\|_F^2}{2N}\times\\
&\qquad \sum_{p=1}^{2N}\left(\|\tilde{\boldsymbol{\chi}_p}\|_F^2 - \frac{\alpha_0 N\|\boldsymbol{A}\|_F}{\|\boldsymbol{A}\|_F^2+\|\boldsymbol{B}\|_F^2}\right)^2 + \text{const.}
\end{aligned} \qquad (20)$$

Next, instead of minimizing (20) with respect to $\{x_m(n)\}_{n=0,m=1}^{N-1,M}$, we resort to the following minimization problem:

$$\begin{aligned}
\min_{\tilde{\boldsymbol{\chi}_p},\boldsymbol{v}_p}\quad & \sum_{p=1}^{2N}\|\tilde{\boldsymbol{\chi}_p}-\boldsymbol{v}_p\|_F^2 \\
\text{s.t.}\quad & |x_m(n)| = 1,\ \|\boldsymbol{v}_p\|_F^2 = \kappa
\end{aligned} \qquad (21)$$

where $\kappa = \dfrac{\alpha_0 N\|\boldsymbol{A}\|_F}{\|\boldsymbol{A}\|_F^2+\|\boldsymbol{B}\|_F^2}$. Without loss of generality, one can see that the criterion in (20) and (21) are "almost equivalent" to each other in the sense that if one takes on a small value, so does the other; particularly, the quadratic terms in (20) become zero if (21) is zero, and vice versa.

To address the minimization problem in (21), we define:

$$\begin{aligned}
\boldsymbol{f}_p &= [e^{-j\omega_p}\ \cdots\ e^{-j2N\omega_p}]^T,\\
\boldsymbol{F} &= [\boldsymbol{f}_1\ \cdots\ \boldsymbol{f}_{2N}],\\
\bar{\boldsymbol{X}} &= [\boldsymbol{X}\ \boldsymbol{0}]_{M\times 2N}^T,\\
\boldsymbol{V} &= [\boldsymbol{v}_1\ \cdots\ \boldsymbol{v}_{2N}]^T.
\end{aligned}$$

Consequently, one can readily rewrite the minimization problem in (21) as

$$\begin{aligned}
\min_{\substack{\{x_m(n)\}_{n=0,m=1}^{N-1,M},\\ \{\boldsymbol{v}_p\}_{p=1}^{2N}}}\quad & \left\|\boldsymbol{F}^H\bar{\boldsymbol{X}} - \boldsymbol{V}\right\|_F^2 \\
\text{s.t.}\quad & |x_m(n)| = 1,\ \|\boldsymbol{v}_p\|_F^2 = \kappa.
\end{aligned} \qquad (22)$$

The criterion in (22) can be efficiently handled via a cyclic minimization approach. For a given $\{x_m(n)\}_{n=0,m=1}^{N-1,M}$, the solution $\{\boldsymbol{v}_p\}_{p=1}^{2N}$ can be given as

$$\boldsymbol{v}_p = \sqrt{\kappa}\frac{\boldsymbol{d}_p}{\|\boldsymbol{d}_p\|_2} \qquad (23)$$

where $\boldsymbol{d}_p^T =$ the p^{th} row of $(\boldsymbol{F}^H\bar{\boldsymbol{X}})$. Furthermore, note that $\left\|\boldsymbol{F}^H\bar{\boldsymbol{X}} - \boldsymbol{V}\right\|_F^2 = \left\|\bar{\boldsymbol{X}} - \boldsymbol{F}\boldsymbol{V}\right\|_F^2$ as $\boldsymbol{F}$ is unitary. Hence, for a given $\{\boldsymbol{v}_p\}_{p=1}^{2N}$, the solution $\{x_m(n)\}_{n=0,m=1}^{N-1,M}$ to (21) can be found as

$$x_m(n) = \exp(j\arg([\boldsymbol{F}\boldsymbol{V}]_{m,n})). \qquad (24)$$

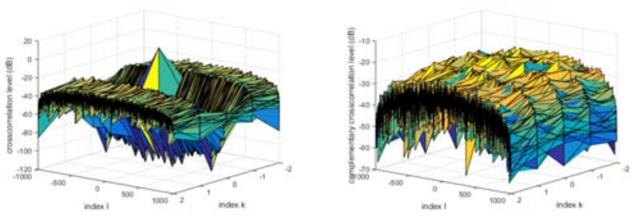
It must be noted that the terms $\boldsymbol{F}^H\bar{\boldsymbol{X}}$ in (23) and $\boldsymbol{F}\boldsymbol{V}$ in (24) are nothing but the FFT of each column of $\bar{\boldsymbol{X}}$ and the IFFT of each column of $\boldsymbol{V}$, respectively. Owing to the nature of these solutions, G-WeCAN is very fast and can essentially be used for generating sequence sets with $N \sim 10^5$ and $M \sim 10^2$.

IV. Generalized CAN from G-WeCAN

Based on above formulations we further introduce a generalized version of the CAN algorithm [11], referred to as (G-CAN). G-CAN is even more computationally efficient than G-WeCAN when, lowering all the out-of-phase correlation and complementary correlation lags has the same importance. Assuming $\{\alpha_n\}_{n=0}^{N-1} = \{\beta_n\}_{n=0}^{N-1} = 1$, the criterion in (6) can be simplified as

$$\begin{aligned}
\tilde{\mathcal{E}} &= \sum_{n=-(N-1)}^{N-1}\|\boldsymbol{R}_n - N\boldsymbol{I}_M\delta_n\|_F^2 + \sum_{n=-(N-1)}^{N-1}\|\boldsymbol{\Gamma}_n\|_F^2 \qquad (25)\\
&= \frac{1}{2N}\sum_{p=1}^{2N}\|\tilde{\boldsymbol{\Phi}}_r(\omega_p) - N\boldsymbol{I}_M\|_F^2 + \|\tilde{\boldsymbol{\Phi}}_\gamma(\omega_p)\|_F^2
\end{aligned}$$

where $\tilde{\boldsymbol{\Phi}}_r(\omega_p) = \sum_n \boldsymbol{R}_n e^{-jn\omega_p}$; $\tilde{\boldsymbol{\Phi}}_\gamma(\omega_p) = \sum_n \boldsymbol{\Gamma}_n e^{-jn\omega_p}$. Furthermore, following the simplification in (17), one can



(a) (b)

Fig. 1. (a) Cross-correlation and (b) complementary cross-correlation levels of the G-WeCAN sequences for $N = 1000, M = 3$.

write $\tilde{\mathbf{\Phi}}_r(\omega) = \boldsymbol{\chi}(\omega)\boldsymbol{\chi}^H(\omega)$ and $\tilde{\mathbf{\Phi}}_\gamma(\omega) = \boldsymbol{\chi}(\omega)\boldsymbol{\chi}^T(\omega)$. Consequently, (25) can be given as

$$\tilde{\mathcal{E}} = \frac{1}{2N}\sum_{p=1}^{2N} \|\boldsymbol{\chi}_p\boldsymbol{\chi}_p^H - N\boldsymbol{I}_M\|_F^2 + \|\boldsymbol{\chi}_p\boldsymbol{\chi}_p^T\|_F^2 \tag{26}$$

which can be further simplified as

$$\begin{aligned}\tilde{\mathcal{E}} &= \frac{1}{2N}\sum_{p=1}^{2N}(2\|\boldsymbol{\chi}_p\|_2^4 - 2N\|\boldsymbol{\chi}_p\|_2^2 + N^2M) \qquad (27)\\ &= N\sum_{p=1}^{2N}\left(\left\|\frac{\boldsymbol{\chi}_p}{\sqrt{N}}\right\|_2^2 - \frac{1}{2}\right)^2 + N^2\left(M - \frac{1}{4}\right).\end{aligned}$$

Finally, using the same argument following (21), the minimization problem can be defined as

$$\begin{aligned}\min_{\boldsymbol{\chi}_p,\tilde{\boldsymbol{v}}_p} \quad & \sum_{p=1}^{2N}\left\|\frac{\boldsymbol{\chi}_p}{\sqrt{N}} - \tilde{\boldsymbol{v}}_p\right\|_2^2 \qquad (28)\\ \text{s.t.} \quad & |x_m(n)| = 1,\ \|\tilde{\boldsymbol{v}}_p\|_2^2 = \frac{1}{2}.\end{aligned}$$

To solve the minimization criterion in (28), we define $\tilde{\boldsymbol{F}} = \frac{1}{\sqrt{N}}[\boldsymbol{f}_1 \ \cdots \ \boldsymbol{f}_{2N}]$. Hence, the minimization problem for G-CAN can simply be given in a matrix form as

$$\begin{aligned}\min_{\bar{\boldsymbol{X}},\tilde{\boldsymbol{V}}} \quad & \left\|\tilde{\boldsymbol{F}}^H\bar{\boldsymbol{X}} - \tilde{\boldsymbol{V}}\right\|_F^2 \qquad (29)\\ \text{s.t.} \quad & |x_m(n)| = 1,\ \|\tilde{\boldsymbol{v}}_p\|_2^2 = \frac{1}{2}.\end{aligned}$$

To find the solutions to (29), one can resort to similar cyclic minimization techniques as detailed in Section III.

V. Numerical Examples

In this section we consider minimizing the criterion in (22) for $N = 1000$ and $M = 3$. We initialize the algorithm with a randomly generated set of sequences with the said length. However, other sequence families with known good correlation properties such as Golay sequences can also be used for initialization. To construct the matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ in (11) that are needed in G-WeCAN, we choose $\alpha_n^2 = \beta_n^2 = 1$ for $n \in [1, 250]$ and zero otherwise. α_0 and β_0 are chosen to make sure that $\boldsymbol{A} \succeq 0$ and $\boldsymbol{B} \succeq 0$. For G-CAN we choose $\alpha_n^2 = \beta_n^2 = 1$ for all n. Fig. 1(a) shows the cross-correlation levels of the constructed G-WeCAN set of sequences for different lags.

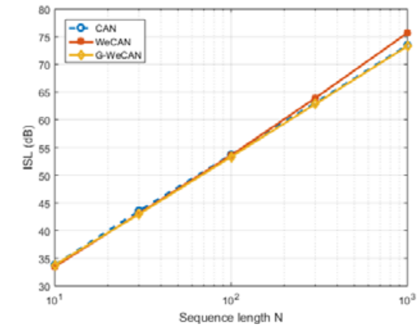


Fig. 2. Comparison of ISL metric for CAN, WeCAN and G-WeCAN sequence with $N = \{10, 30, 100, 300, 1000\}$ and $M = 3$.

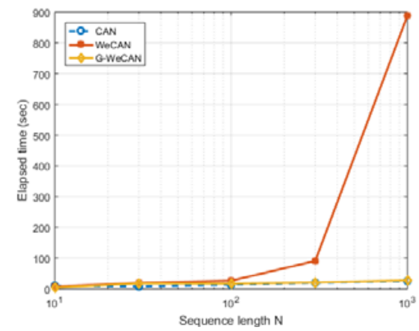


Fig. 3. Comparison of computation times for CAN, WeCAN and G-WeCAN sequence with $N = \{10, 30, 100, 300, 1000\}$ and $M = 3$.

We also compare the generalized algorithm with previously suggested CAN and WeCAN algorithms in terms of overall ISL metric for different sequence length. We generate sets of sequences with sequence length $N = \{10, 30, 100, 300, 1000\}$ and $M = 3$ for CAN, WeCAN and G-WeCAN. Fig. 2 depicts that G-WeCAN shows better performance than its CAN and WeCAN counterparts. It can be noted that, whereas WeCAN requires $2N$ computations of SVD of an $N \times M$ matrix, G-WeCAN relies on computations of FFT coefficients. Due to this fact, G-WeCAN is much faster than WeCAN and can be essentially used for generating sequences with length in the order of 10^5. Fig. 3 shows the comparison of required computation times for CAN, WeCAN and G-WeCAN sequences with $N = \{10, 30, 100, 300, 1000\}$ and $M = 3$ on a standard PC.

VI. Conclusion

We presented novel cyclic algorithms, referred to as G-WeCAN and G-CAN, to minimize a generalized WISL criterion to design sets of unimodular sequences that have good correlation and complementary correlation properties. A number of numerical examples were provided to demonstrate the good correlation and complementary correlation properties of the sets of unimodular sequences obtained by the proposed algorithms.

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