Limits of Transmit Beamforming for Massive MIMO Radar

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Beamforming in Massive MIMO

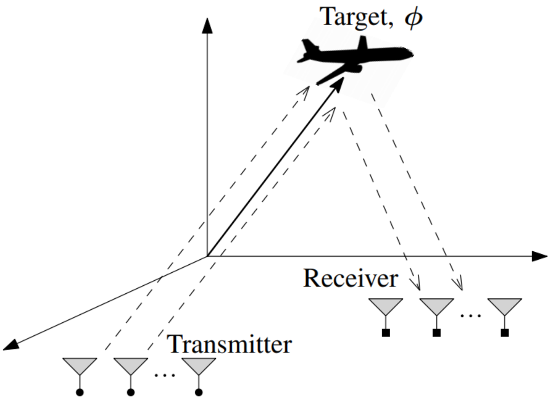

Figure: Co-located MIMO radar.

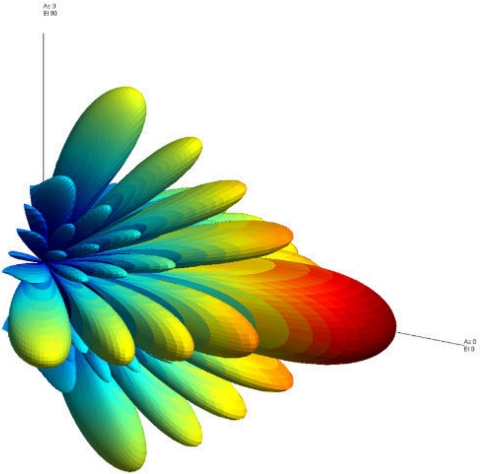

Figure: Typical beampattern in a radar system.

Questions

As the number of antennas, N, grows large

1. What beampatterns can be realized if the covariance matrix of the transmit signals may be chosen at will?
2. How rapidly we can change the beampattern for closely located angles?
3. How our ability to form a peak in a beampattern is governed by the number of antennas?

Formulation

An array of N Tx antennas transmitting $x_n(l) \in \mathbb{C}$
The base band signal:

$$\sum_{n=1}^{N} e^{-j2\pi f_0 \tau_n(\theta)} x_n(l) \triangleq \mathbf{a}^H(\theta)\mathbf{x}(l), \quad l \in \{1, 2, \cdots, L\}$$

The power of the probing signal (transmit beampattern) at θ:

$$p(\theta) = \mathbf{a}^H(\theta)\mathbf{R}\mathbf{a}(\theta)$$

where $\mathbf{R} = \mathbb{E}\{\mathbf{x}(l)\mathbf{x}^H(l)\}$ is the signal covariance matrix.
Assume $\tau_n(\theta) = n\theta$ for ULA and $\xi = 2\pi f_0 = 1$,

$$p(\theta) = \sum_{k=1}^{N}\sum_{l=1}^{N} R_{k,l} e^{j(k-l)\theta}$$

Limits of Beamforming

Note that

A covariance matrix $\mathbf{R}$ of size $N \times N$ can always be realized with N independent streams of signals, transmitted by N antennas.

Realization and Resolution (l_0)

Q: What functions $p(\theta)$ can be realized using N antennas, if the covariance matrix $\mathbf{R} \in \mathbb{C}^{N \times N}$ may be chosen at will?

Definition

The Zero-Order Resolution, $l_0(N)$ is defined as the number of points in space for which we can exactly determine the power, i.e., we can design the covariance matrix of the signal transmitted by N antennas in order to achieve the allocated power.

The Finite-Energy Case

When $\|\mathbf{R}\|_F$ is bounded. Note that

$$p(\theta) = \mathbf{a}^H(\theta)\mathbf{R}\mathbf{a}(\theta) = \mathrm{tr}(\mathbf{R}\mathbf{a}(\theta)\mathbf{a}^H(\theta)) = \mathrm{tr}(\mathbf{R}\bar{\mathbf{A}}(\theta))$$

Let $\mathbf{E} = \bar{\mathbf{A}}(\theta_2) - \bar{\mathbf{A}}(\theta_1)$ for θ_1 and θ_2. Then

1. $|p(\theta_2) - p(\theta_1)| = |\mathrm{tr}(\mathbf{RE})|$ will be small for a small $|\theta_2 - \theta_1|$
2. Given a smoothness of $p(\theta)$, an N^2 point realization of the beampattern is achievable.

The Unconstrained-Energy Case

The psd constraint can be equivalently expressed as $\mathbf{R} = \mathbf{X}^H\mathbf{X}$ for any $\mathbf{X} \in \mathbb{C}^{N\times N}$. Then $\|\mathbf{X}\mathbf{a}(\theta)\|_2 = \sqrt{p(\theta)}$. Suppose the beam pattern $p(\theta)$ is to be realized at N locations $\{\theta_k\}_{k=1}^N$ and thus $\mathbf{X}\mathbf{a}(\theta_k) = \sqrt{p(\theta_k)}\mathbf{u}_k$. Let

$$\begin{aligned}
\mathbf{A} &= [\mathbf{a}(\theta_1)\ \mathbf{a}(\theta_2)\ \cdots\ \mathbf{a}(\theta_N)], \\
\mathbf{U} &= [\mathbf{u}_1\ \mathbf{u}_2\ \cdots\ \mathbf{u}_N], \\
\mathbf{D} &= \mathrm{Diag}\left(\left[\sqrt{p(\theta_1)}\ \sqrt{p(\theta_2)}\ \cdots\ \sqrt{p(\theta_N)}\right]\right).
\end{aligned}$$

Note that $\mathbf{A}$ is a non-singular Vandermonde matrix. Then

$$\begin{aligned}
& \mathbf{X}\mathbf{A} &&= \mathbf{U}\mathbf{D} \\
\Rightarrow\quad & \mathbf{X} &&= \mathbf{U}\mathbf{D}\mathbf{A}^{-1} \\
\Rightarrow\quad & \mathbf{R} &&= \mathbf{A}^{-H}\mathbf{D}\mathbf{U}^H\mathbf{U}\mathbf{D}\mathbf{A}^{-1}
\end{aligned}$$

Rate of Innovation (l_1)

Q: How rapidly we can change the beampattern, $p(\theta)$ for closely located angles θ using N antennas?

Note that,

$$\frac{\partial p(\theta)}{\partial \theta} = \sum_{k,l} j(k-l) R_{k,l}\, e^{j(k-l)\theta},$$

which implies

$$\left| \frac{\partial p(\theta)}{\partial \theta} \right| \leq 2 \sum_{k>l} (k-l)\, |R_{k,l}| \triangleq l_1(\mathbf{R})$$

Rate of Innovation (I_1) (contd.)

$$\frac{|I_1(\mathbf{R})|^2}{\|\mathbf{R}\|_F^2} \leq \frac{\alpha}{6} N^2(N^2 - 1)$$

where $\alpha = \frac{\|\mathbf{R}\|_F^2 - R_{\text{diag}}}{\|\mathbf{R}\|_F^2}$.

Theorem

Assuming that the transmission power is fixed with respect to the number of antennas, N, the rate of innovation $I_1(N)$ behaves as $\mathcal{O}(N^2)$ with respect to N.

Forming a Peak (l_2)

Q: How our ability to form a peak in a beampattern is governed by the number of antennas?

To form a peak one must be able to make the second derivative of $p(\theta)$ “large”:

$$\frac{\partial^2 p(\theta)}{\partial \theta^2} = \sum_{k,l} -(k-l)^2 R_{k,l}\, e^{j(k-l)\theta}$$

which implies that

$$\left| \frac{\partial^2 p(\theta)}{\partial \theta} \right| \leq 2 \sum_{k>l} (k-l)^2 \left| R_{k,l} \right| \triangleq l_2(\mathbf{R})$$

Forming a Peak (I_2) (contd.)

It follows from above

$$\frac{|I_2(\mathbf{R})|^2}{\|\mathbf{R}\|_F^2} = \mathcal{O}(N^6)$$

Theorem

Assuming that the transmission power is fixed with respect to the number of antennas, N, forming of a peak in the beampattern, $I_2(N)$ behaves as $\mathcal{O}(N^3)$.

Numerical Studies

Mathematically, the beampattern matching is accomplished by solving the following problem,

$$\begin{aligned} \min_{\zeta, \boldsymbol{R}} \ & \frac{1}{K}\sum_{k=1}^{K}\omega_k[\boldsymbol{a}^H(\theta_k)\boldsymbol{R}\boldsymbol{a}(\theta_k) - \zeta d(\theta_k)]^2 \\ \text{s.t.} \ & R_{n,n} = c/N, \ n = 1, \cdots, N, \\ & \boldsymbol{R} \succeq \boldsymbol{0} \end{aligned}$$

where

$$d(\theta) = \begin{cases} 1, & \theta \in [\tilde{\theta}_k - \frac{\triangle}{2}, \tilde{\theta}_k + \frac{\triangle}{2}], \quad k = 1, 2, 3, \cdots \\ 0, & \text{otherwise} \end{cases}$$

Numerical Studies (contd.)

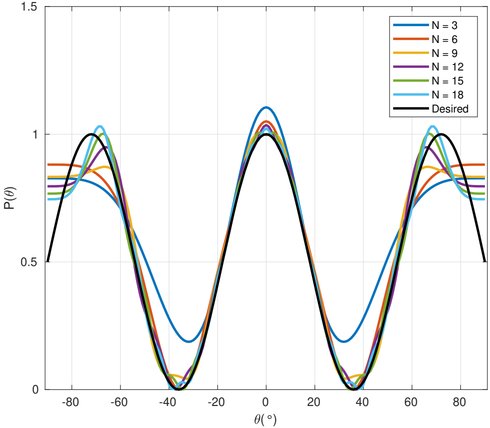

Figure: Realization of a smooth sinusoidal beampattern with zero-order resolution $K = 181$ using $N \in \{3, 6, 9, 12, 15, 18\}$ antennas.

Numerical Studies (contd.)

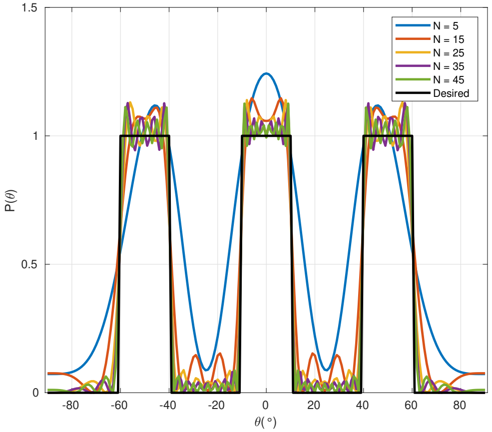

Figure: Realization of a rectangular beampattern with resolution $K = 181$ using $N \in \{5, 15, 25, 35, 45\}$ antennas.

Numerical Studies (contd.)

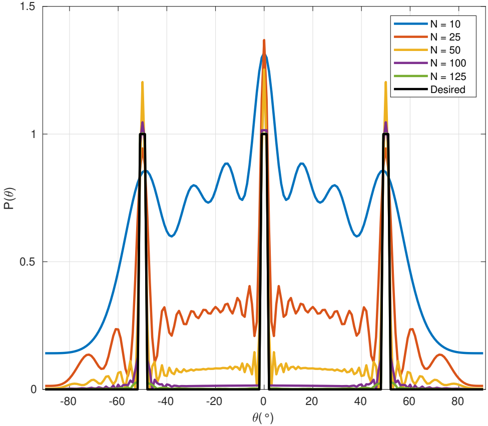

Figure: Realization of a impulse-like beampattern with resolution $K = 181$ using $N \in \{10, 25, 50, 100, 125\}$ antennas.

Summary

- The fundamental limitations of the resolution of beampatterns produced by MIMO radars in relation to their number of antennas.
- Multiple analytical results to show how the changes in a beampattern are impacted by an increased number of antennas in a massive MIMO scenario.
- Future research: The characterization and efficient construction of such beampatterns.

Thank you
and
Questions?

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