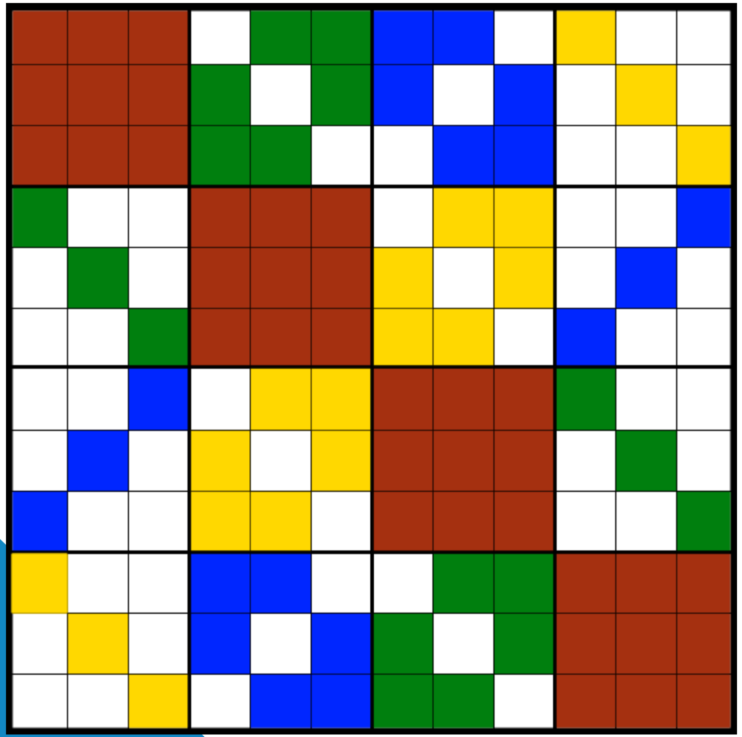
WaveOPT Lab

EFFICIENT CONSTRUCTION OF POLYPHASE SEQUENCES WITH OPTIMAL PEAK SIDELOBE LEVEL GROWTH



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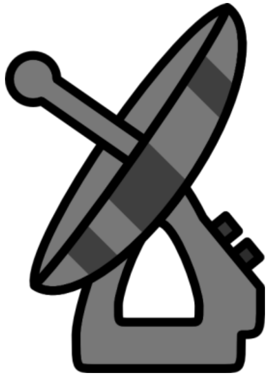
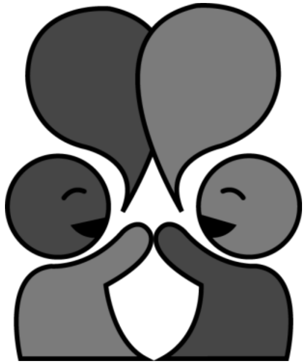
Applications



- Communication systems
- Active Sensing
- Channel estimation
- Synchronization



Motif

"Designing analytical tool to construct polyphase sequences with optimal PSL growth"

Preliminaries

- Good correlation properties
- Sequences

$$\boldsymbol{x} = [x(1)\ x(2)\ \cdots\ x(N)]^T, \qquad x(n) = e^{2\pi i n/K}$$

- Auto-correlations:

$$\text{aperiodic:} \quad r(k) \triangleq \sum_{l=1}^{N-k} x(l)x^*(l+k) = r^*(-k), \quad k \in \{0, \cdots, N-1\}$$

$$\text{periodic:} \quad c(k) \triangleq \sum_{l=1}^{N} x(l)x^*(l+k)_{mod\ N}, \qquad k \in \{0, \cdots, N-1\}$$

Good Correlation Properties: Metrics

$$\begin{aligned}
\text{peak sidelobe level:} \quad & \mathrm{PSL}(\boldsymbol{x}) && = \max\{|r(k)|\}_{k=1}^{N-1}, \\
\text{integrated sidelobe level:} \quad & \mathrm{ISL}(\boldsymbol{x}) && = \sum_{k=1}^{N-1} |r(k)|^2, \\
\text{merit factor:} \quad & \mathrm{MF}(\boldsymbol{x}) && = \frac{|r_0|^2}{2\sum_{k=1}^{N-1} |r(k)|^2} = \frac{\mathrm{E}^2}{2\mathrm{ISL}}.
\end{aligned}$$

Earlier Results: Peak Side-lobe Level (PSL)

- Asymptotic behavior: $m(N) = \min_{\boldsymbol{x} \in \mathcal{X}_N} \mathrm{PSL}(\boldsymbol{x})$
- Earlier Results:

$$\begin{aligned} m(N) &\leq 1 \quad \forall\, N \leq 5 \\ m(N) &\leq 2 \quad \forall\, N \leq 21 \\ m(N) &\leq 3 \quad \forall\, N \leq 48 \\ m(N) &\leq 4 \quad \forall\, N \leq 82 \\ m(N) &\leq 5 \quad \forall\, N \leq 105 \end{aligned}$$

Earlier Results (contd.):

Conjecture: As $N \to \infty$, we have $\dfrac{m(N)}{\sqrt{N}} \to d$, where $d = 0.435...$

- Moon and Moser ['68]

$$m(N) \leq (2+\epsilon)\sqrt{N \log N}$$

- Mercer ['06]

$$m(N) \leq (\sqrt{2}+\epsilon)\sqrt{N \log N}$$

Sequence Sets

Let $X = \{\boldsymbol{x}_m\}_{m=1}^{M}$ be a subset of sequence of length N with $\|\boldsymbol{x}_m\|_2^2 = N, \forall\ m$, having optimal PSL growth.

- Aperiodic correlation:

$$r_{X;pq}(k) \triangleq \sum_{l=k+1}^{N} x_p(l) x_q^*(l-k) = r_{X;pq}^*(-k)$$

$$p, q \in \{1, \cdots, M\}, \qquad k \in \{0, \cdots, N-1\}.$$

Correlation Matrix:

$$\boldsymbol{R}_{X;k} = \begin{bmatrix} r_{X;11}(k) & r_{X;21}(k) & \dots & r_{X;M1}(k) \\ r_{X;12}(k) & r_{X;22}(k) & \dots & r_{X;M2}(k) \\ \vdots & \vdots & \ddots & \vdots \\ r_{X;1M}(k) & r_{X;2M}(k) & \dots & r_{X;MM}(k) \end{bmatrix}$$

$$k = -N+1, \cdots, 0, \cdots, N-1.$$

$$\boldsymbol{J}_k = \begin{bmatrix} \overbrace{0 \cdots 1}^{k+1} & & 0 \\ & \ddots & \\ & & 1 \\ 0 & & \end{bmatrix}^T_{N \times N} = \boldsymbol{J}^T_{-k}, \qquad k = 0, \cdots, N-1$$

$$\boldsymbol{R}_{X;k} = X^* \boldsymbol{J}_k X = \boldsymbol{R}^*_{X;-k}, \qquad k = 0, \cdots, N-1.$$

Metrics for Sequence Sets:

- PSL: $$\mathcal{P}_X \triangleq \max(|r_{X;pq}(k)|_{p\neq q;k} \cup |r_{X;pp}(k)|_{p;k\neq 0}),$$

- ISL: $$\mathcal{I}_X \triangleq \sum_{p\neq q;k} |r_{X;pq}(k)|^2 + \sum_{p;k\neq 0} |r_{X;pp}(k)|^2,$$

$$p, q \in \{1, \cdots, M\}, k \in 0, \cdots, N-1.$$

Bounds:

Welch PSL lower bound: $$B_{\mathcal{P}_X} \triangleq N\sqrt{\frac{M-1}{2NM-M-1}},$$

Welch ISL lower bound: $$B_{\mathcal{I}_X} \triangleq N^2 M(M-1).$$

$$\mathcal{P}_X \sim \sqrt{\frac{M-1}{2M}}\,\sqrt{N},$$

$$\mathcal{P}_X \lesssim \frac{1}{\sqrt{2}}\,\sqrt{N}.$$

Proposed Approach:

Let $\boldsymbol{s}$ be a sequence with entries $\{s(n)\}_{n=1}^{N}$. We design $\boldsymbol{s}$ as a linear combination of the sequence set $\boldsymbol{X}$ and a weighted multiplier $\boldsymbol{\phi}$,

$$\boldsymbol{s} = \sum_{m=1}^{M} \phi(m)\boldsymbol{x}_m = \boldsymbol{X}\boldsymbol{\phi}$$

Proposed Approach (cont.)

$$\begin{aligned} r_{\boldsymbol{s}}(k) &\triangleq \sum_{l=k+1}^{N} s(l)s^*(l-k) \\ &= \sum_{p=1}^{M}\sum_{q=1}^{M}\left(\phi(p)\phi^*(q)\sum_{l=k+1}^{N} x_p(l)x_q^*(l-k)\right) \end{aligned}$$

$$\begin{aligned} |r_{\boldsymbol{s}}(k)| &\leq \sum_{p=1}^{M}\sum_{q=1}^{M} |\phi(p)|\,|\phi^*(q)|\,|r_{X,pq}(k)| \\ &\leq \max_{p,q}\{|r_{X,pq}(k)|\}\left(\sum_{p=1}^{M}\sum_{q=1}^{M}|\phi(p)||\phi^*(q)|\right) \\ &\leq \mathcal{P}_{\boldsymbol{X}}\ \|\boldsymbol{\phi}\|_1^2. \end{aligned}$$

Proposed Approach!!

$$\mathcal{P}_s \lesssim \frac{\|\phi\|_1^2}{\sqrt{2}} \sqrt{N}.$$

Optimal Construction

$$\min_{\{s(n)\}_{n=1}^{N};\{\phi(m)\}_{m=1}^{M}} \quad f = \|\boldsymbol{X}\boldsymbol{\phi} - \boldsymbol{s}\|_2^2$$

$$\text{s.t.} \quad \boldsymbol{s} \in \Omega$$

$$\boldsymbol{\phi} = [\phi(1)\ \phi(2)\ \cdots\ \phi(M)]^T,$$

$$\boldsymbol{s} = [s(1)\ s(2)\ \cdots\ s(N)]^T,$$

Cyclic Minimization

$$\text{For fixed } \boldsymbol{s} \; : \widehat{\boldsymbol{\phi}} = \boldsymbol{X}^{+}\boldsymbol{s},$$

$$\text{For fixed } \phi \; : \widehat{\boldsymbol{s}} = \text{SGN}(\Re(\boldsymbol{X}\boldsymbol{\phi}))$$

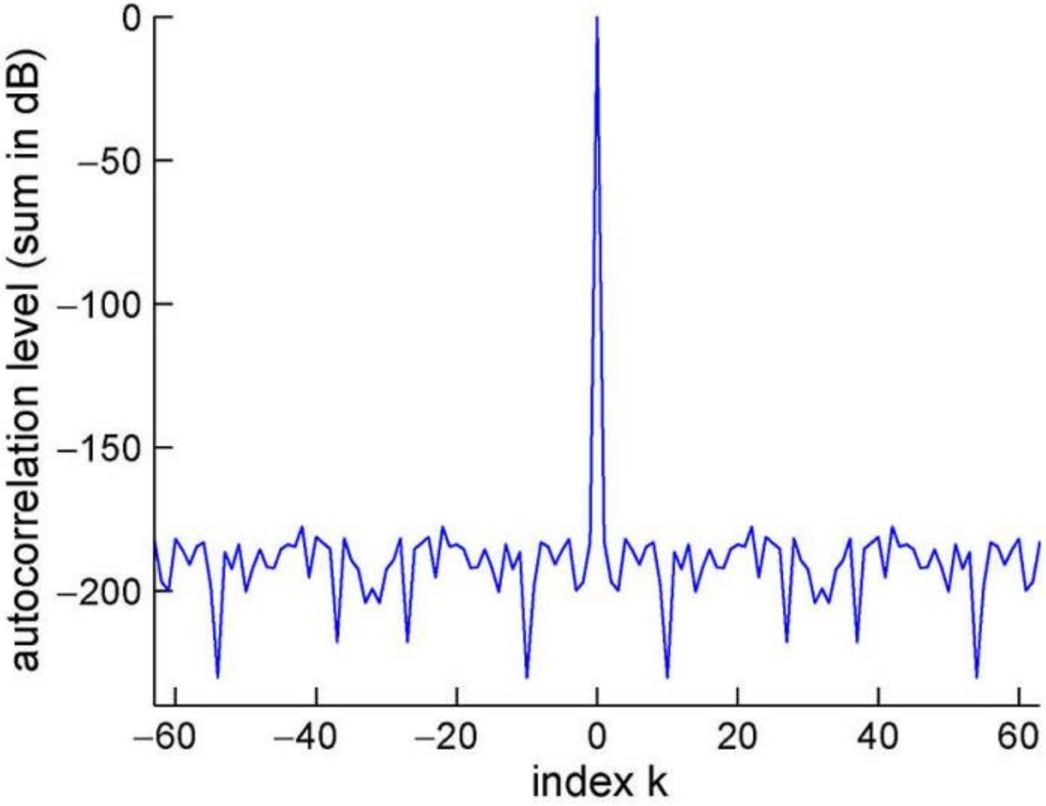
Direct Optimization

$$\min_{\{s(n)\}_{n=1}^{N}} \|\boldsymbol{X}\boldsymbol{X}^{+}\boldsymbol{s}-\boldsymbol{s}\|_2^2$$

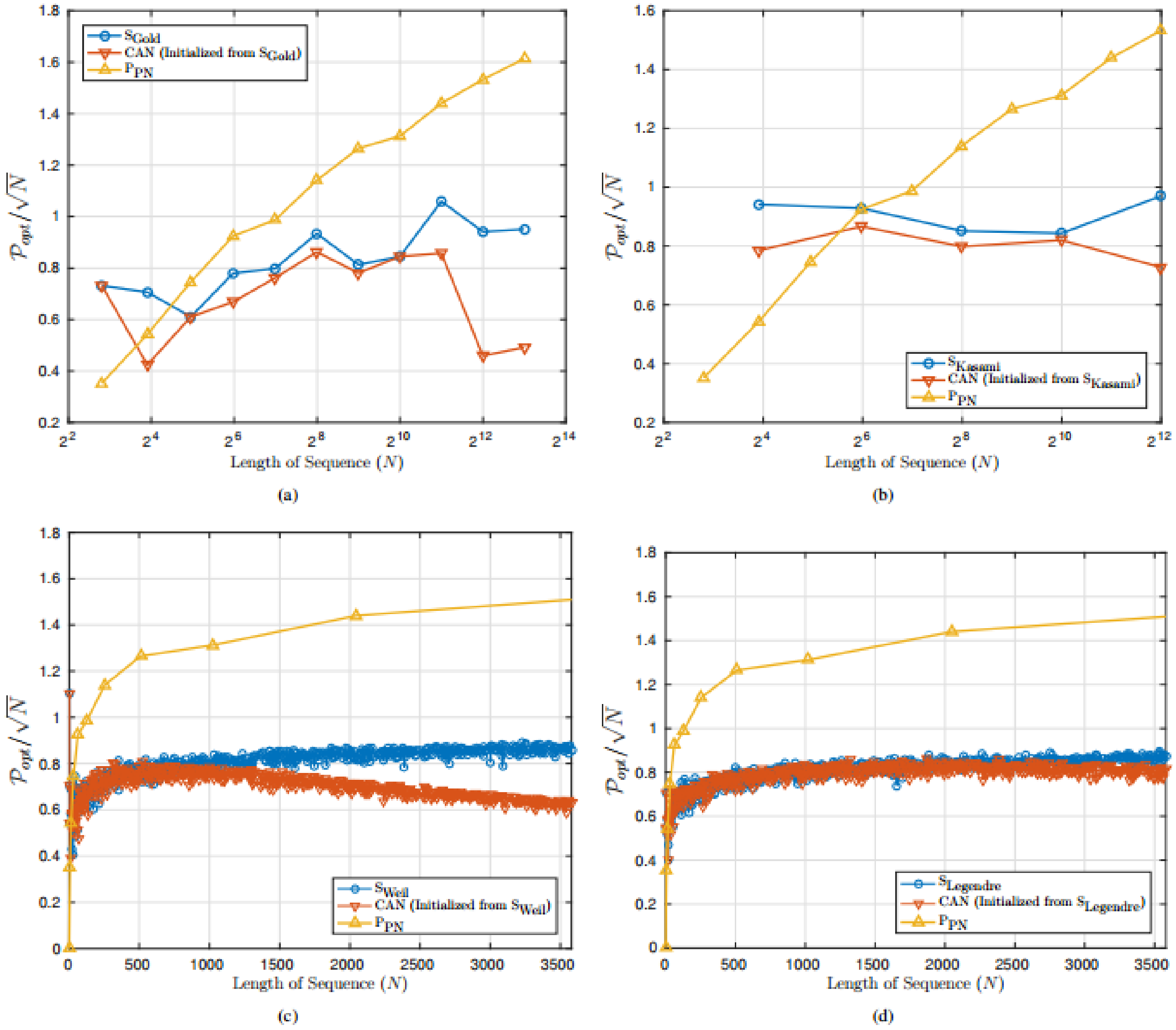
$$\begin{aligned}
&\|\boldsymbol{X}\boldsymbol{X}^{+}\boldsymbol{s}-\boldsymbol{s}\|_2^2\\
&=(\boldsymbol{X}\boldsymbol{X}^{+}\boldsymbol{s}-\boldsymbol{s})^*(\boldsymbol{X}\boldsymbol{X}^{+}\boldsymbol{s}-\boldsymbol{s})\\
&=\boldsymbol{s}^*\boldsymbol{X}\boldsymbol{X}^{+}\boldsymbol{X}\boldsymbol{X}^{+}\boldsymbol{s}\ +\ \boldsymbol{s}^*\boldsymbol{s}\ -\ 2\boldsymbol{s}^*\boldsymbol{X}\boldsymbol{X}^{+}\boldsymbol{s}\\
&=\ -\boldsymbol{s}^*\boldsymbol{X}\boldsymbol{X}^{+}\boldsymbol{s}\ +\ N.
\end{aligned}$$

$$\boldsymbol{s}_{\text{opt}}=\arg\max_{\boldsymbol{s}\in\Omega}\ \boldsymbol{s}^*\boldsymbol{X}\boldsymbol{X}^{+}\boldsymbol{s}$$

Numerical Examples



Numerical Examples (cont.)



Thank You

Questions?